HYSTERESIS EFFECT IN REPAIR AND REPLACEMENT PROBLEMS WITH DOWNTIME

By

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To my loving wife, Marci,

and to my parents

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HYSTERESIS EFFECT IN REPAIR AND REPLACEMENT
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A new class of repair and replacement problems called the hysteresis problem is investigated. This problem is concerned with the amount of recovery of service life following a repair with arbitrary time distribution, extreme cases of which are minimal and major repair. Incorporated in the cost structure of the hysteresis problem are variable operating costs, repair costs, replacement costs, and loss of revenue per unit of downtime. A mathematical formulation for the discounted hysteresis problem is presented using the functional equation approach. Optimal age replacement policies, which minimize the long run total expected discounted cost, are examined for the linear hysteresis problem.

The extreme case of major repair is investigated, and a generalization of previously obtained results is determined. For the linear hysteresis problem, specific cost functions are examined for an equipment subject to a constant failure rate. In addition, numerical results for the optimal age replacement policy are obtained. An extension of the linear hysteresis problem incorporating an

expenditure repair function to control the amount of recovery of service life is examined. A new operating policy, which is an extension of the age replacement policy, is investigated numerically for different age recovery functions. A renewal process associated with the linear hysteresis problem is examined, and a probability mass function obtained.

CHAPTER 1

THE PROBLEM

1.1 Introduction

The literature concerned with the determination of optimal replacement policies for continuously aging equipment subject to failure and continuous review has examined only the problems of minimal and major repair. The distinction between these problems is in the 'amount' of recovery of equipment life following a repair. Whereas no recovery of equipment life occurs after a minimal repair, there is a complete recovery of equipment life following a major repair. Recovering equipment life is equivalent to reducing service age, the total operating time of the equipment.

For most of the literature, the costs associated with repair and replacement have been considered invariant with respect to the service age of the equipment. In addition, the majority of the literature has been concerned with the determination of necessary and sufficient conditions for minimizing either total expected discounted cost or expected cost per unit time over an infinite time horizon. Not until the last two years has any attention been focused on continuous variable operating and repair costs. Also, only a few papers in the literature have assumed that repairs are not instantaneous.

Between the extremes of minimal repair and major repair exists the situation of partial repair. Following this repair there is a hysteresis effect, which is a partial reduction of service age. This is in contrast to a complete (no) reduction of service age in the case of a major (minimal) repair. The amount of partial reduction (or

hysteresis) can be a function of the length of the repair, the expense incurred in undertaking the repair, and the service age of the equipment at the instant of failure. In addition, by varying the amount of partial reduction of service age, it is possible to recover the extreme problems of minimal and major repair. The problem generated by considering the amount of partial reduction of service age shall be called the hysteresis problem. To date, optimal policies have not been determined for the hysteresis problem.

A special case of the hysteresis problem, called the linear hysteresis problem, occurs when the function defining the partial reduction of service age is linear. If the amount of partial repair in this problem is economically controlled, the resulting problem shall be called the economically controlled linear hysteresis problem.

This dissertation presents the formulation of the hysteresis problem using an optimization approach due to Bellman [3] (also Bellman and Dreyfus [4]), called the functional equation approach. Analytic and quantitative results are obtained for the linear hysteresis problem. For the economically controlled linear hysteresis problem, quantitative results are obtained for which a qualitative analysis is performed. Some stochastic elements of the hysteresis problem are also investigated.

The remainder of this chapter is devoted to several developmental aspects of the hysteresis problem. There is a discussion of the basic optimization criteria considered in the literature and in this dissertation. An extensive literature survey is presented. The topics covered include: minimal repair problem, major repair problem,

preventive maintenance hysteresis-type problem, and downtime for repair. The functional equation approach is introduced and discussed as a means of providing a background for the formulation of the hysteresis problem. This formulation yields a functional-differential equation, the solution to which is determined for the major repair problem and the linear hysteresis problem with equipment failure rate function equal to a constant in Chapters 2 and 3, respectively. The functional-differential equation for the economically controlled linear hysteresis problem is also developed. This chapter concludes with an outline of the remaining chapters.

1.2 Optimization Criteria

Two approaches to replacement theory, known as the renewal theoretic approach and the functional equation approach, result in different optimization criteria. For equipment subject to failure, the renewal theoretic approach and the functional equation approach result in the criterion of expected cost per unit time and the criterion of total expected discounted cost, respectively. These criteria apply for either a finite or infinite time span. When equipment is not subject to failure and a discount factor is assumed, the only incentive to replace, if ever, is the increasing cost disadvantage of retaining the present equipment versus replacing it with a new equipment. This is known as the classical equipment replacement problem. For this problem, the functional equation approach yields the criterion of total discounted cost.

For equipment subject to failure, the difference between the optimization criteria is a consequence of the method of aggregating

the relevant costs over a given length of time. In the renewal theoretic approach, the focus is on the expected number of occurrences (or renewals) of two or more events over a given length of time. As a result of this approach, many interesting statistics concerning the stochastic process of service age in the repair-replacement problem are obtained. Since associated with each occurrence of an event is a cost, expressions for the expected cost for a given length of time and the expected cost per unit time are readily obtained. In contrast, the functional equation approach accumulates future expected costs in a cost function. Unless these future costs are discounted to the present (concept of present value), the cost function would grow without bound for an infinite time span. For a finite time span, it is still important to consider discounting since future alternatives are being examined in terms of present day values. As a result, expressions such as total expected discounted cost for either a finite or infinite time span arise.

When equipment is not subject to failure, the element of uncertainty is removed from the optimization criterion. This results in a criterion of total discounted cost for the functional equation approach. The functional equation approach will be discussed in greater detail in Section 1-4.

1.3 Literature Review

The literature of equipment repair and replacement has differentiated according to the amount of effort expended on the repair of a failed equipment. The amount of repair can take on the range of possibilities from minimal repair to major repair or overhaul. Minimal

repair usually involves the repair or replacement of a minor part of the equipment, sufficient to return the equipment to an operational status as soon as possible. An example of minimal repair is the replacement of a blown fuse in an electrical piece of equipment. A major repair (or overhaul) usually involves the intense testing and repairing and/or replacing of parts of the equipment. The status of the equipment after major repair is considered to be as good as new. Due to the expense of the initial investment and/or the size of the equipment, overhauls are usually limited to equipment which is never replaced. For this reason and because major repairs return equipment to an as good as new state, major repairs are equivalent to replacement. Examples of equipment which are subject to major repair are jet engines, electric generators, and diesel engines. For the remainder of this discussion, it is assumed that only failure induces a repair, whether minimal, partial, or major.

A failed equipment which is returned to an operational status after a minimal repair has the same service age after the repair as prior to the repair. Thus, the equipment failure rate function remains unchanged by minimal repair. When the equipment service age attains a predetermined value, the equipment is replaced with an identical new one. This type of repair and replacement problem shall be referred to as Problem 1 or the minimal repair problem. After a failed equipment undergoes a major repair, the service age of the equipment is that of a new equipment, which is zero. Thus after major repair, the failure rate function is that of a new piece of equipment. When the equipment attains a predetermined service age, it is replaced with an identical

new equipment. This type of repair and replacement problem shall be referred to as Problem II or the major repair problem.

An extension to Problems I and II is a repair and replacement problem which lies between these problems. Repairing a failed equipment beyond a minimal repair but not up to a major repair is often feasible. Minimal repair often entails the partial disassembly of the equipment. Instead of simply repairing or replacing the failed part, it may be economically feasible to repair or replace parts which have not failed, since the initial labor cost for disassembling the equipment has already been paid. After such a repair, the service age of the equipment is less than its service age prior to commencing the repair. The partial reduction of service age can be a function of repair costs, length of downtime, and service age at the instant of failure. In addition, replacements are made with identical new equipment. This type of repair and replacement problem shall be referred to as Problem III or the hysteresis problem. It can even be rationalized that a minimal repair results in some recovery of service age.

In order to determine when to repair or replace an equipment, it is necessary to utilize an inspection policy. This policy is used to determine the state, either operating or failed, and the service age of the equipment. Inspection policies can be continuous or discrete. Continuous inspection policies involve constant monitoring of the equipment. Discrete inspection policies are customarily periodic although some random policies have been examined in the literature.

Equipment aging can be either discrete or continuous. Discrete

aging is often associated with a periodic inspection policy in which the inspection determines the state of the equipment, where the state can have a value from 0 to S inclusive. The state 0 denotes a failed equipment and the state S denotes a new equipment. After inspection, the equipment is classified into one of S + 1 states and a decision is made to either (a) replace, or (b) perform preventive maintenance, or (c) do nothing. If the equipment can be classified into S + 1 operational states, transitions from one state to other states are specified by some probability mass function, and periodic inspections take place, then the resulting model is one of Markovian Replacement. Continuous aging is associated with both discrete and continuous inspection, although the latter is more common in replacement theory.

The literature of replacement theory will be surveyed for Problems I, II, and III. Under the different optimization criteria previously discussed, the authors in replacement theory have considered primarily the following costs and parameters. They are

K = fixed cost incurred for planned replacement;

k = instantaneous repair cost incurred at failure;

i = interest rate;

F( ) = lifetime distribution function of the equipment,

λ( ) = failure rate function or hazard function.

The failure rate function $\lambda(\cdot)$ is also referred to in the literature as the failure rate. The last section of the literature review covers demerits for repair.

a. Problem I - minimal repair

The earliest studies in the area of classical equipment

replacement were concer[illegible] comparing an [illegible] a potential replacement. In 1949, Terborgh [24] pr[illegible] for comparing a challenger (new equipment) and identical future challengers with a defender (present equipment). Grant [14] was concerned with equipment obsolescence, increasing maintenance cost, and decreasing output (increasing equipment inefficiency). He presented procedures for determining the time to replacement of an operating equipment when more efficient equipment was available, no discount factor was assumed, and, the operations cost was non-decreasing. In conjunction with the Machinery and Allied Products Institute [17], Terborgh extended his earlier work to incorporate the concepts of capital investment. This resulted in the development of the classical MAPI model.

Bellman [5] (also Bellman and Dreyfus [6]) demonstrated a method of approach to problems of replacement with an identical equipment and with a superior equipment (technologically superior). The method used a dynamic programming formulation to set up a discrete time functional equation. In order to determine a replacement age X for the equipment, Bellman's objective function was the maximization of the long run total discounted return (profit). Bellman's model incorporated the concepts of salvage value, increasing operations cost, decreasing output of the equipment, and discount factor 1. Failure of the equipment was not considered, and the only incentive to replace the equipment was the increasing cost disadvantage of retaining the equipment versus replacing it with a new equipment. Descamps [10] solved the continuous time version of Bellman's functional equation for

[illegible]

Employing [illegible] renewal theory [illegible] [illegible] considered the problem of scheduling replacement of [illegible] failure rate $\lambda(u)$, and the fixed costs k and K for a minimal repair and replacement, respectively. Minimal repair and replacement were [illegible] to be instantaneous. Further, the failure rate at the termination of the minimal repair was the same as at the commencement of the [illegible]. Using a criterion of minimizing the long run total expected [illegible] per unit time, Barlow and Hunter determined that a unique optimal [illegible] replacement age X existed as the solution to

$$\int_0^X u\lambda'(u)du = \frac{K}{k}$$

[illegible] the failure rate was strictly increasing to infinity. [illegible] also considered an age replacement problem, which will [illegible] in the following section.

In 1973 Sivazlian [21] extended Barlow and Hunter's result for minimal repair, which had assumed instantaneous minimal repair, to include an arbitrary distribution of downtime for repair. Incorporated in the model were a variable repair cost k(u), a variable operations cost c(u), a cost for revenue lost per unit of downtime r(u), a failure rate $\lambda(u)$, and a discount factor i. In addition, a general distribution function of repair times was incorporated. A functional equation approach, similar to that of Descamps [10] and Beckmann [4] was used to minimize the long run total expected discounted cost

[illegible] determined that a necessary and sufficient condition for the [illegible]ence of a unique optimal equipment replacement age X, where X is [illegible] unique solution to the equation

$$\int_0^X [i - e^{-J(w)}]dc(w) = X$$

[illegible] c(w) $(w \geq 0)$ be strictly increasing to infinity Now

$$J(w) = \int_0^w j(x)dx$$

[illegible]

$$j(x) = i + \lambda(x)[1 - \overline{K}_2(i)]$$

and

$$r(w) = \frac{i}{j(w)} \{c(w) + \frac{\pi(w)}{i} [j(w) - i] + \lambda(w)k(w)\}$$

where $\overline{K}_2(i)$ is the Laplace-Stieltjes Transform with parameter i of the general distribution function of repair times Due to the structure of c(w), it was not necessary for the failure rate to be strictly increasing to infinity for a unique optimal equipment replacement age to exist

b Problem II - major repair

Every major repair problem in the literature has considered two fundamental costs a fixed cost K for a planned replacement, and a fixed cost h for a major repair or replacement at failure. In addition, it has been assumed that a planned replacement costs less than or equal to a repair or replacement at failure ($h \geq 0$, $0 < K \leq h$) This is logical

[illegible] unscheduled repair or replacement would involve an unplanned [illegible] in the use of the equipment, and the possible activation of [illegible] piece of equipment which would not be as efficient. The majority of the papers in the area of major repair for a continuously aging piece of equipment subject to failure and continuous inspection have concluded that a sufficient condition for a unique finite optimal equipment replacement age X to exist is that the failure rate be continuous and strictly increasing to infinity.

Considering only the costs k and K and the distribution function F, Barlow and Hunter [1] (also Barlow and Proschan [2], [3]) examine the major repair problem with instantaneous major repair and planned replacement. Following a method of objective function formulation (Morse [18]) using renewal theory, Barlow and Hunter constructed an objective function for the average cost per unit time over a finite time span, which was the weighted sum (by the costs, K and k) of the expected number of planned replacements and the expected number of unplanned replacements (or major repair) per unit time over the time span. Then letting the finite time span approach infinity, Barlow and Hunter minimized the long run expected cost per unit time to obtain X as the solution to

$$\lambda(X) \int_0^X \bar{F}(t)dt - F(X) = \frac{K}{k - K}$$

X will be finite and unique if the failure rate $\lambda(u)$ is continuous and strictly increasing to infinity. The optimal age for a planned replacement is infinite if either (1) the failure rate $\lambda(u) = \lambda$ = constant, or (2) the cost of a planned replacement is equivalent to the cost of a

major repair (or unplanned replacement). The latter is a consequence of it never being economically advantageous to replace prior to failure.

Specific results for the unique equipment replacement age for Barlow and Hunter's [1] major repair problem were obtained by Glasser [13] for the truncated normal, Gamma, and Weibull distributions. For each distribution, a graph was presented from which the unique optimal replacement age is readily ascertained after the computation of a cost ratio.

Fox [12] investigated the effects of incorporating a discount factor in the Barlow and Hunter's major repair problem. Continuous discounting was assumed, and the total loss was equal to the sum of the discounted losses incurred at each planned replacement and failure. He determined that provided the failure rate (for known distributions) was continuous and strictly increasing to infinity, a unique equipment replacement age X exists which minimizes the long run total expected discounted loss (or cost). Denardo and Fox [7] proved that even if the cost of a replacement at failure exceeded the cost of a planned replacement, it is not optimal to replace an operating equipment during periods of decreasing failure rate.

Beckmann [4] approached the same problem studied by Fox [12] by formulating a functional equation in a manner analogous to that of Bellman [5] and Descamps [10]. His objective function minimized the long run total expected discounted cost. A necessary condition for a unique optimal equipment replacement age X to exist was determined, but was not presented in a closed form expression.

Employing renewal theory, Scheaffer [20] extended Barlow and

Hunter's [1] major repair problem to include an increasing equipment operations cost ct^{a} $(a > 0)$. A cost for operation related to equipment age is reasonable, and intuitively appealing, since as an equipment ages the output may decrease or the equipment may become more inefficient. By minimizing the long run expected cost per unit time, Schaffer determined that a finite equipment replacement age X exists which is the unique solution to

$$[(k - K)\lambda(X) + acX^{a-1}] \int_0^X \bar{F}(t)dt - (k - K)F(X)$$

$$- c[X^{a}\bar{F}(X) + \int_0^X t^{a}\, dF(t)] = K$$

provided that the failure rate is continuous and non-decreasing. The incorporation of an increasing equipment operations cost has altered the condition determined by Barlow and Hunter that the failure rate be strictly increasing to infinity in order that the equipment replacement age X will be finite and unique. For the exponential distribution function, it was demonstrated that the unique equipment replacement age is finite provided that the operations cost is strictly increasing to infinity. For a constant or decreasing operations cost, an infinite replacement age is optimal for the exponential distribution function.

The area of Markovian Replacement was pioneered by Derman [8]. Under the assumption of Markov Chains with stationary transition probabilities, Derman developed three theorems, which proved that for an equipment subject to Markovian deterioration with costs associated with the observed state (i.e., determined by inspection) and the decision made (replace or do nothing), the optimal stopping rules belong

to a subclass of all possible rules Further, the optimal stopping rules can be determined by solving linear programming problems Derman's objective function for his results was the long run total expected cost

Extensions to Derman's work to obtain formal rules based upon the costs for repair at failure, replacement, and the observed state have been made by Derman [9], Kolesar [16], and Ross [19]

c. Problem III - hysteresis

Only one paper, and that in the area of Markovian Replacement, has considered the hysteresis effect Eppen [11] determined conditions under which a policy for preventive maintenance is optimal for a Markovian deteriorating system with a discrete state space Following each inspection to determine in which one of the $S + 1$ states (states are $0, \ldots, S$) the system is, a decision is made to either leave the system in the present state or place it in a newer and higher state by performing some maintenance action Since the system deteriorates from S to 0, it improves in the reverse direction It is assumed that all maintenance actions are performed instantaneously The cost of improving the system by z states (z integer) is cz ($c > 0$) For each state an operating cost $L(\ )$ is assigned, such that $L(\ )$ is a finite, positive, real valued function defined on $0, 1, \ldots, S$ which possesses the properties; (1) $L(\cdot)$ is convex and (2) $\Delta L < - c^2$ The deterioration of the system is described by one-step-down transition matrices A functional equation approach was employed to minimize the cost of operating and maintaining the system The conditions for the policy of preventive maintenance were based on the one-step-down transition matrices

[illegible] Downtime for Repair

[illegible] [22], [23] has investigated [illegible] processes associated with operating and repair times [illegible] that the succession of operating times between failures is independently and identically distributed, as is the succession of repair times. Moreover, the operating and repair times are mutually independent. The operating and repair states form disjoint sets, such that the stochastic process of system state (operating or repair) takes on a value in each set in turn. Given the successive sojourn times in the two sets, Takata determined the long run asymptotic distribution of the sum of the sojourn times spent in a repair state. Special cases for the sojourn times of operating and repair times were discussed.

Weiss [25] formulated an approach for incorporating repair times (or downtimes) in the major repair problem. He assumed that the distribution of repair times was arbitrary and the only cost incurred in connection with the repair was the fixed cost of the repair (i.e., no cost per unit of downtime). Conditions were determined under which an optimal policy for the major repair problem did not necessarily exist.

Table 1.1 summarizes the significant papers for Problems I and II.

1.4 Functional Equation Approach

In this section, an optimization approach known as the functional equation approach is examined. This approach, which is fundamental to dynamic programming, is utilized in the development of the hysteresis

[illegible] Significant Papers for Problems I and II

[illegible]	Minimal Repair	Major Repair	Downtime for Repair	Replacement Cost	Repair Cost	Operating Cost	Discounting	[illegible]	[illegible]	[illegible]
Dollman [5]	X	—	No	V	—	V	Yes			[illegible]
[illegible]escamps [10]	X		No	V	—	V	Yes	—		[illegible]
Barlow and Hunter [1]	X		No	F	F	No	No	Yes		[illegible]
[illegible]orlian [21]	X	—	Yes	V	V	V	Yes	[illegible]		[illegible]
Barlow and Hunter [1]	—	X	No	F	F	No	No	Yes		[illegible]
Fox [12]	—	X	No	F	F	No	Yes	[illegible]		[illegible]
Beckmann [4]		X	No	F	F	No	Yes	[illegible]		[illegible]
Scheaffer [20]		X	No	F	F	V	No	[illegible]		[illegible]

V - variable
F - fixed
D - discrete
C - continuous

FE - functional equation
RT - renewal theory

[illegible] following section. As applied to replacement [illegible] the functional equation approach examines, for each [illegible] service age, the costs associated with each of the alternati[illegible] replace and do not replace, and selects the one which is smal[illegible]. It is assumed that the cost curve associated with the altern[illegible] not replace' intersects at least once the 'cost curve associat[illegible] the alternative replace', where the curves are a function of [illegible] service age. It therefore follows that alternating decision [illegible] or do not replace and replace exist. Since it is assumed th[illegible] equipment has an initial age less than or equal to X, then o[illegible] age X at which the first region of do not replace terminates [illegible] interest.

The different decision regions including the point(s) [illegible] indifference are illustrated by the following application of the functional equation approach. Assume continuous aging. Define for $x \geq 0$:

- $h(x)$ = total expected discounted cost over an infinite [illegible] horizon for an equipment aged x when following [illegible] optimal policy,
- $K(x)$ = replacement cost at age x incorporating a salvage feature,
- $f(x)$ = minimum total expected discounted cost over an in[illegible] time horizon.

The principle of optimality of dynamic programming states that [illegible] respect to the current decision of replace or do not replace, [illegible] future decisions are optimal. Then the expected cost of a dec[illegible] to replace at age x is $K(x) + h(0)$. Further if no replacement [illegible] at age x, the expected cost for this decision is $h(x)$. Then if [illegible]

age at which the equipment is replaced, the following fun[illegible]
equation is obtained

$$f(x) = \min \begin{cases} K(x) + h(0) & x = X \text{ (r[illegible]} \\ h(x) & 0 \le x < X \text{ do [illegible]} \end{cases}$$

Several conclusions concerning decision region [illegible] of $f(x)$ at $x = X$ can be drawn from this functional equation [illegible] age x a replace policy is preferred to a do not replace p[illegible] and only if

$$K(x) + h(0) < h(x)$$

Similarly, a do not replace policy at age x is preferred [illegible] policy if and only if

$$K(x) + h(0) > h(x)$$

These two conclusions generate the decision regions of do [illegible] and replace. The cost curves and the alternating decision [illegible] are illustrated in Figure 1.1. At $x = X$, the decision mak[illegible] indifferent to either a policy of replace or do not replace [illegible] only if

$$K(x) + h(0) = h(x)$$

For this last conclusion to be true, it is necessary tha[illegible] $f(x)$ be continuous at $x = X$.

Continuity of the function $f(x)$ at $x = X$ implies tha[illegible] of $f(x)$ as x approaches X from the left and the limit of $f(x)$ [illegible] x approaches X from the right are equal. Therefore the cost [illegible]

[illegible] and replace $\lambda(x + [illegible]$ [illegible]

[illegible]ty of the function $f(x)$ at $x =$ [illegible] opposite, that is a discontinu[illegible] [illegible] following paradox occurs [illegible] a discontinuity (i e [illegible]) the cost curve correspond[illegible] not replace is lower than the cost curve of replace [illegible] that the equipment should not yet be replaced. To the [illegible] discontinuity (i e , $x > X$), the cost curve corresponding [illegible] lower than the cost curve of do not replace, which implie[illegible] equipment should already have been replaced. To resolve th[illegible] it is necessary to require continuity of the function $f(x)$ [illegible]. Thus a replacement occurs at age X

1.5 Problem Formulation

a. The Functional Equation for the Hysteresis Probl[illegible]

It is assumed that the equipment is continuously moni[illegible] reviewed and that failures are detected instantly. Further, th[illegible] planning horizon is infinite and all replacements are made wit[illegible] identical equipment. All replacements are assumed to occur ins[illegible] following a failure, there is a length of time called the downti[illegible] which the equipment is repaired before being returned to an operati[illegible] status. A zero downtime corresponds to an instantaneous repair. [illegible] the termination of the repair, the service age of the equip[illegible] governed by the hysteresis age function $s(x)$, where x is the se[illegible] age of the equipment at the instant of failure and $s(x)$ is the [illegible] age at the termination of the repair

The [illegible] that $\alpha(x)$ can assume [illegible] this corresponds to no reduction [illegible] is the minimal repair problem [illegible] reduction of service age to [illegible] which is the major repair problem [illegible] $0 \leq \alpha(x) \leq x$, a partial reduction, or hysteresis of service age occurs, which is the hysteresis problem (Problem III). Figures 1.2, 1.3, and 1.4 are sample functions of the service age process $S(t)$ for Problems I, II, and III with downtime for repair, respectively. Thus the hysteresis age function covers Problems I, II, and III. The linear hysteresis problem where $\alpha(x) = ax$ for $0 \leq a \leq 1$ will be investigated in this dissertation.

For a specified value of the replacement age X ($X > 0$), the following replacement policy is known as an X-policy. Given an equipment with service age u, it is retained if $0 \leq u < X$ and replaced instantly if $X \leq u < \infty$, where all replacements are made with identical new (service age zero) equipment in an operating state. Define for $u \geq 0$

$K(u)$ = replacement cost at age u incorporating a salvage feature (non recurring cost),

$k(u)$ = repair cost associated with failure at age u and incurred instantaneously (i.e., at commencement of repair),

$c(u)du$ = operating cost of equipment between u and $u + du$ (recurring cost),

$e(u)$ = loss of revenue per unit of downtime for an equipment with age u and in a state of repair,

$\lambda(u)du$ = probability that the equipment will fail between ages u and $u + du$ given that the equipment is in an operating state at age u,

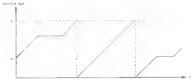

Figure 1.2 Sample function of the service age process $S(t)$ for the minimal repair problem with downtime, where x is the initial equipment age, and X is the equipment replacement age.

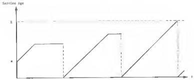

Figure 1.3. Sample function of the service age process $S(t)$ for the major repair problem with downtime, where x is the initial equipment age, and X is the equipment replacement age.

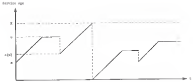

Figure 1.4. Sample function of the service age process $S(t)$ for the hysteresis problem with downtime, where x is the initial equipment age, X is the equipment replacement age, and $a(u)$ is the hysteresis age function

$a(u)$ = hysteresis age function for an equipment aged u where $a(u)$ is the age of the equipment at the end of a repair,

i = interest rate expressed over continuous time, where exponential discounting is employed,

$f_1(u)$ = minimum total expected discounted cost over an infinite time horizon when the equipment has age u and is in operation,

$f_0(u)$ = minimum total expected discounted cost over an infinite time horizon when repair on a failed equipment aged u has just commenced.

It is assumed that the functions $K(u)$, $k(u)$, $c(u)$, $r(u)$, and $\lambda(u)$ are all continuous and $K(u)$ is bounded from above.

A functional equation describing the replacement decision of the X-policy in a manner analogous to Beckmann [4] and Descamps [10] is now formulated. Since equipment ages continuously in time while operating and is currently in an operating state with present age of x ($0 \le x \le X$), the following functional equation is obtained

$$f_1(x) = \min \begin{cases} K(x) + f_1(0) & x = X \text{ (replace)} \\ c(x)dx + (1-idx)[1 - \lambda(x)dx]f_1(x+dx) & \\ \quad + (1-idx)\,\lambda(x)dx\, f_0(x+dx) & \\ \quad + o(dx) & 0 \le x < X \text{ (do not replace)} \end{cases} \qquad (1.1)$$

where $o(dx)$ is defined by $\lim\limits_{dx \to 0} \dfrac{o(dx)}{dx} = 0$.

The first alternative in the functional equation (1.1) corresponds to a replacement action with an associated fixed cost $K(x)$ and a cost of commencing with a new equipment $f_1(0)$. The second alternative corresponds to a retain action, thereby incurring an equipment operations

and the sum of the discounted costs associated with the equipment in an operating state and age $x + dx$ and the equipment having failed at age $x + dx$ and repair commencing

For $0 \leq x < X$, no replacement occurs and $f_1(x)$ satisfies the relation

$$f_1(x) = e(x)dx + (1 - idx)[1 - \lambda(x)dx]f_1(x + dx)$$
$$+ (1 - idx)\,\lambda(x)dx\, f_0(x + dx) + o(dx) \qquad (1\ 2)$$

Before proceeding to form a Taylor series expansion of $f_0(x + dx)$ and $f_1(x + dx)$, it is necessary to state some assumptions concerning these functions. It is assumed that the properties of $f_0(x)$ and $f_1(x)$ are continuity, monotone increasing, and existence of the first derivative. Then performing a Taylor series expansion on $f_0(x + dx)$ and $f_1(x + dx)$, where the remainder term (after the first two terms in the expansion) is of order $o(dx)$ as dx approaches zero, the following is obtained

$$f_1(x) = e(x)dx + (1 - idx)[1 - \lambda(x)dx][f_1(x) + f_1'(x)dx]$$
$$+ (1 - idx)\lambda(x)dx[f_0(x) + f_0'(x)dx] + o(dx) \qquad (1\ 3)$$

Simplifying (1 3) yields

$$f_1(x) = e(x)dx + f_1(x) + f_1'(x)dx - [i + \lambda(x)]f_1(x)dx$$
$$+ \lambda(x)f_0(x)dx + o(dx) \qquad (1\ 4)$$

Rearranging (1 4), dividing the result by dx and letting $dx \to 0$, the following is obtained

$$f_1'(x) - [i + \lambda(x)]f_1(x) + \lambda(x)f_0(x) = -\, e(x) \qquad (1\ 5)$$

Before solving equation (1.5), it is necessary to obtain an expression for $f_0(x)$ in terms of $f_1(a(x))$ using a derivation due to [illegible]zaliz [21]. Assume that an equipment is in an operating state and has a service age u such that $0 \leq u < X$. If the equipment fails between u and u + du, repair commences instantaneously. The length of downtime, or repair time, T is a random variable whose distribution function $W_T(\cdot)$ is not dependent on the failure characteristics of the equipment. It is assumed that the succession of downtimes and service times forms a sequence of mutually independent random variables (Takacs [23]). At the conclusion of the repair, the service age of the equipment is $a(u)$ $(0 \leq a(u) \leq u)$, where $a(u)$ is the hysteresis age function. From the characteristics of $a(u)$, it follows that the service age is reduced due to the completion of a repair whenever $a(u) < u$.

For an equipment aged u which has just failed, the conditional long run total expected discounted cost is

$$k(u) + \int_0^t r(u)e^{-i\theta}d\theta + f_1(a(u))e^{-it} \tag{1.6}$$

given that the repair time has length t and terminates at time t at which time the equipment is returned to an operating state with a service age $a(u)$. Thus the unconditional long run total expected discounted cost for an equipment aged u which has just failed is

$$f_0(u) = \int_0^\infty [k(u) + \int_0^t r(u)e^{-i\theta}d\theta + f_1(a(u))e^{-it}]dW_T(t)$$

$$= k(u)\int_0^\infty dW_T(t) + \frac{r(u)}{i}\int_0^\infty (1 - e^{-it})dW_T(t)$$

$$+ f_1(a(u)) \int_0^\infty e^{-it} dH_T(t)$$

$$= k(u) + \frac{r(u)}{i} + [f_1(a(u)) - \frac{r(u)}{i}] \int_0^\infty e^{-it} dH_T(t) \qquad (1.7)$$

Let

$$\overline{h}_T(s) = \int_0^\infty e^{-st} dH_T(t) , \qquad \text{Re } s \geq 0 \qquad (1.8)$$

denote the Laplace-Stieltjes transform of the repair time distribution function $H_T(\cdot)$, which exists provided that $H_T(t)$ is sectionally continuous in every finite interval $0 \leq t \leq M$ and of exponential order β for $t > M$. Upon using (1.8) in (1.7), the following is obtained

$$f_0(u) = k(u) + \frac{r(u)}{i} + [f_1(a(u)) - \frac{r(u)}{i}]\overline{h}_T(i)$$

Using the last expression to substitute for $f_0(x)$ in (1.5) and rearranging yields

$$f_1'(x) - [i + \lambda(x)]f_1(x) + \lambda(x)\overline{h}_T(i)f_1(a(x))$$

$$= - c(x) - k(x)\lambda(x) - \frac{r(x)}{i} [1 - \overline{h}_T(i)]\lambda(x) \qquad (1.9)$$

Let

$$g(x) = c(x) + k(x)\lambda(x) + \frac{r(x)}{i} [1 - \overline{h}_T(i)]\lambda(x)$$

Substituting the prior expression in (1.9) results in the functional-differential equation

$$f_1'(x) - [i + \lambda(x)]f_1(x) + \lambda(x)\overline{h}_T(i)f_1(a(x)) = - g(x) \qquad (1.10)$$

When $a(x) = x$ (minimal repair) or $a(x) = 0$ (major repair), the

solution to (1.10) can be determined by using an appropriate integrating factor. In Chapter 2, the solution to (1.10) with $a(x) = 0$ is obtained and investigated. A general analytic solution to the functional-differential equation (1.10) does not appear to be available or obtainable (Appendix A). However, it is possible to derive a solution expression in the special case when the hysteresis age function is linear and the failure rate function is equal to a constant. This special case is investigated in Chapter 3. Recall that when the hysteresis age function is linear (i.e., $a(x) = ax$), the resulting problem is the linear hysteresis problem.

b. The Economically Controlled Linear Hysteresis Problem

Consider the linear hysteresis problem with an added refinement: it is desired to control economically the amount of partial repair and the corresponding fractional reduction of service age at the termination of the repair by spending funds at the start of the repair. This expenditure is in addition to the repair cost $k(\cdot)$ incurred at the commencement of the repair and is a function of the level of expenditure E and the age u at which the repair commences. The function so defined is $C(E,u)$, where

$$C(E,u) = Eu^m \quad , \quad m = 0,1,2,\ldots \tag{1.11}$$

If no funds are expended, that is E equals zero, then a minimal repair will be performed.

The service age at the termination of the repair is a function of the level of expenditure E and the age u at which the repair commences. If $C(E,u)$ is spent on the repair, then the fraction of

[illegible] is $p(E)$, where $0 \leq p(E) \leq 1$, [illegible] the service age at the end of the repair is $p(E)u$. The new problem generated by considering the new functions $C(E,u)$ and $p(E)$ shall be called the economically controlled linear hysteresis problem.

The value of E acts as a decision variable since it controls both $p(E)$ and $C(E,u)$. Consequently, the previously defined X-policy is modified to incorporate E, and becomes a two-variable decision policy. For specified values of the level of expenditure E $(E \geq 0)$ and the replacement age X $(X > 0)$, the following is known as the E,X-policy. An equipment with service age u is retained if $0 \leq u < X$. Further if failure has just occurred, $C(E,u)$ is spent on repair to reduce to $p(E)u$ the service age of the equipment. If $X \leq u < \infty$, then the equipment is replaced instantaneously, where all replacements are made with identical new (service age zero) equipment in an operating state. Thus, the E,X-policy determines when to replace the equipment and the service age following a repair.

Under the added assumptions of the economically controlled linear hysteresis problem, the conditional long run total expected discounted cost for an equipment aged u $(u < X)$ which has just failed is

$$C(E,u) + k(u) + \int_0^t r(u)e^{-i\theta}d\theta + f_1(p(E)u)e^{-it} \tag{1.12}$$

given that the repair time has length t and terminates at time t at which time the equipment is returned to an operating state with a service age $p(E)u$. Noting the similarity of (1.12) and (1.6), it follows from examining (1.8) that the unconditional long run total expected discounted cost for an equipment aged u which has just failed is

$$f_0(x) = C(E,c) + k(c) + \frac{r(c)}{i} + \left[f_1(p(E)c) - \frac{r(c)}{i}\right]\overline{H}_T(i)$$

Using the last expression to substitute for $f_0(x)$ in (1.5) and rearranging yields

$$f_1'(x) - [i + \lambda(x)]f_1(x) + \lambda(x)\overline{H}_T(i)f_1(p(E)x)$$

$$= - c(x) - k(x)\lambda(x) - \frac{r(x)}{i}[1 - \overline{H}_T(i)]\lambda(x) - C(E,x)\lambda(x) \tag{1.13}$$

Let

$$q(x) = c(x) + k(x)\lambda(x) + \frac{r(x)}{i}[1 - \overline{H}_T(i)]\lambda(x) + C(E,x)\lambda(x)$$

Substituting the prior expression in (1.13) results in the functional-differential equation

$$f_1'(x) - [i + \lambda(x)]f_1(x) + \lambda(x)\overline{H}_T(i)f_1(p(E)x) = - q(x) \tag{1.14}$$

for the economically controlled linear hysteresis problem. For the case when the failure rate function is equal to a constant, the solution to the functional-differential equation (1.14) is determined in Chapter 4. The E,X-policy is investigated quantitatively in Chapter 4 using this solution.

1.6 Chapter Outline

In Chapter 2, results obtained by Barlow and Hunter, Fox, Beckmann, and Scheaffer for the major repair problem are extended to incorporate the concept of downtime for repair. Starting with the functional-differential equation (1.10) with $a(x) = 0$, a sufficient

[illegible] a unique finite optimal equipment replacement age X to exist [illegible] derived for age replacement with major repair and downtime. Several corollaries to this condition are also presented. An expression is obtained for the undiscounted long run expected cost per unit time.

The linear hysteresis problem with downtime for partial repair and constant failure rate function is investigated in Chapter 3. An nth degree polynomial with all positive coefficients is assumed for the cost function g(x), and the functional-differential equation (1.10) is solved for a(x) = ax. Three particular polynomial forms are investigated, and sufficient conditions are derived for a unique finite equipment replacement age X to exist for the linear hysteresis problem.

In Chapter 4 the results of Chapter 3 are applied to study the economically controlled linear hysteresis problem with downtime for partial repair. For several curves p(X), two expenditure repair functions are examined: constant and linear. Quantitative results are obtained instead of analytic results, due to the structure of the solution of the functional-differential equation (1.14) for each of the expenditure repair and the curve p(X) functions.

Chapter 5 examines the renewal process associated with the number of operating periods between successive replacements. A probability mass function is determined for the number of operating periods in [0,X]. Using this result, the expected value of the number of operating periods in [0,X] is obtained.

Conclusions and recommendations for future research are presented in Chapter 6.

CHAPTER 2

AGE REPLACEMENT WITH MAJOR REPAIR AND DOWNTIME

2.1 Introduction

Previously obtained results for the minimal repair problem are extended in this chapter. These results, in the form of conditions for the existence of an optimal age replacement policy, called an R-policy, were obtained by Barlow and Hunter [1], Fox [12], Beckmann [4], and Schaeffer [20]. They considered the concepts of fixed repair and replacement costs ([1], [12], [4], [20]), discounting ([12], [4]), and variable operating cost ([20]).

Additional concepts are incorporated in the investigation of the major repair problem in this chapter. They are variable repair cost, downtime for major repair following a failure, and loss of revenue per unit of downtime. Using the criterion of long run total expected discounted cost, the major repair problem with downtime is investigated utilizing an approach due to Beckmann [4]. A sufficient condition for a unique finite optimal equipment replacement age X to exist is determined for the major repair problem with downtime. Four corollaries to this condition are presented. In addition, an expression for the undiscounted long run expected cost per unit time is determined from $f_x(0)$, the minimum total expected discounted cost over an infinite time horizon when the equipment is as good as new (service age zero) and is in operation.

2.2 A Sufficient Condition for a Unique Replacement Age X

The functional-differential equation (1.16) of Chapter 1 is

recalled It is

$$f_1'(x) - [i + \lambda(x)]f_1(x) + \lambda(x)\bar{h}_I(i)f_1(a(x)) = -g(x) \qquad (2\ 1)$$

where

$$g(x) = c(x) + k(x)\lambda(x) + \frac{r(x)}{i}[1 - \bar{h}_I(i)]\lambda(x) \qquad (2\ 2)$$

The procedure will be to solve the functional-differential equation (2 1) for the major repair problem by using an appropriate integrating factor. The boundary condition from the alternative of replace in (1 1), that is

$$f_1(X) = K(X) + f_1(0)$$

is then used to solve for $f_1(0)$. For major repair, $a(x) = 0$, and substituting this expression in (2 1) yields

$$f_1'(x) - [i + \lambda(x)]f_1(x) = -g(x) - f_1(0)\bar{h}_I(i)\lambda(x) \qquad (2\ 3)$$

Now

$$R(u) = e^{-\int_0^u \lambda(\theta)d\theta}, \quad u \geq 0$$

where $R(u)$, the reliability function, is the probability that an equipment of age 0 will survive age u. Note that $R(u)$ is continuous and $R(0) = 1$. Then let

$$\bar{R}(u) = e^{-iu}R(u), \quad u \geq 0 \qquad (2\ 4)$$

and therefore

$$d[\bar{R}(u)] = -[i + \lambda(u)]\bar{R}(u)du \qquad (2\ 5)$$

Note that since $R(0) = 1$, then by (2 4) $\bar{R}(0) = 1$

Multiplying both sides of (2-3) by $\bar{R}(x)$ and integrating yields

[illegible] $f_1(x)$

$$f_1(x) = \frac{f_1(0)}{\bar{R}(x)} - \frac{f_1(0)\bar{h}_T(i)}{\bar{R}(x)} \int_0^x \lambda(u)\bar{R}(u)du$$

$$- \frac{1}{\bar{R}(x)} \int_0^x g(u)\bar{R}(u)du \tag{2.6}$$

If an optimal X-policy exists, continuity of the function [illegible](x) at $x = X$ is required. It then follows that

$$f_1(X) = K(X) + f_1(0) \tag{2.7}$$

Using (2-7) in (2.6), it follows that at $x = X$

$$K(X) + f_1(0) = \frac{f_1(0)}{\bar{R}(X)} - \frac{f_1(0)\bar{h}_T(i)}{\bar{R}(X)} \int_0^X \lambda(u)\bar{R}(u)du$$

$$- \frac{1}{\bar{R}(X)} \int_0^X g(u)\bar{R}(u)du$$

Solving for $f_1(0)$

$$f_1(0) = \frac{- K(X)\bar{R}(X) - \int_0^X g(u)\bar{R}(u)du}{\bar{R}(X) - 1 + \bar{h}_T(i) \int_0^X \lambda(u)\bar{R}(u)du}$$

or

$$f_1(0) = \frac{K(X)\bar{R}(X) + \int_0^X g(u)\bar{R}(u)du}{1 - \bar{R}(X) - \bar{h}_T(i) \int_0^X \lambda(u)\bar{R}(u)du} \tag{2.8}$$

[illegible] (2.5) and $\bar{R}(0) = 1$ in (2.8), the following relation is obtained

$$
\begin{aligned}
(1 + R(X) - \overline{h}_I(t)) \int_0^X \lambda(u)\bar{R}(u)du
&= -\int_0^X d[\bar{R}(u)] - \overline{h}_I(t)\int_0^X \lambda(u)\bar{R}(u)du \\
&= \int_0^X [1 + \lambda(u)]\bar{R}(u)du - \overline{h}_I(t)\int_0^X \lambda(u)\bar{R}(u)du \\
&= \int_0^X \{1 + \lambda(u)[1 - \overline{h}_I(t)]\}\bar{R}(u)du
\end{aligned}
$$

Let

$$v(u) = 1 + \lambda(u)[1 - \overline{h}_I(t)] , \quad u \geq 0 \tag{2.9}$$

This is the expression that Bivalian [31] defined for $j(u)$ in the minimal repair problem (see 1.3a). Then

$$1 - R(X) - \overline{h}_I(t)\int_0^X \lambda(u)\bar{R}(u)du = \int_0^X v(u)\bar{R}(u)du \tag{2.10}$$

Substituting (2.10) in (2.8) yields

$$f_1(0) = \frac{K(X)R(X) + \int_0^X g(u)\bar{R}(u)du}{\int_0^X v(u)\bar{R}(u)du}$$

For the remainder of this chapter, the analysis is restricted to the case where the fixed cost of replacement $K(X)$ is independent of the replacement age X (i.e., no salvage factors incorporated in the replacement cost). Since $K(X) = K$ = constant, then

$$f_1(0) = \frac{K\bar{R}(X) + \int_0^X g(u)\bar{R}(u)du}{\int_0^X v(u)\bar{R}(u)du} \qquad (2\ 11)$$

A central theorem for age replacement with major repair and downtime is now presented

<u>Theorem 2 1</u>

If $\frac{g(u) - K\lambda(u) - Ki}{v(u)}$ $(u \geq 0)$ is strictly increasing to infinity, then a unique finite optimal equipment replacement age X exists which is the solution to the equation

$$\int_0^X v(u)\left[\frac{g(X) - K\lambda(X) - Ki}{v(X)} - \frac{g(u) - K\lambda(u) - Ki}{v(u)}\right]\bar{R}(u)du = K$$

and if the initial age, x, satisfies $x \leq X$ then an optimal X-policy exists for the major repair problem with arbitrary repair time distribution

<u>Proof</u>

If the initial equipment age x is such that $x < X$, then it is clear from substituting (2 11) into (2 5) that if an optimal replacement age X exists, it minimizes both $f_1(0)$ and $f_1(x)$ for all x Differentiating (2 11) with respect to X and equating the result to zero yields

$$[K\hat{R}'(X) + g(X)\bar{R}(X)]\int_0^X v(u)\bar{R}(u)du$$
$$- v(X)\bar{R}(X)[K\bar{R}(X) + \int_0^X g(u)\bar{R}(u)du] = 0 \qquad (2\text{-}13)$$

where

$$R'(X) = \frac{d[R(X)]}{dX}$$

Rearranging (2.13) and dividing the result by $v(X)R(X)\int_0^X v(u)R(u)du$ yields

$$\frac{KR(X) + \int_0^X g(u)R(u)du}{\int_0^X v(u)R(u)du} = \frac{KR'(X) + g(X)R(X)}{v(X)R(X)}$$

Applying (2.5) in the last expression, the following is obtained

$$\frac{KR(X) + \int_0^X g(u)R(u)du}{\int_0^X v(u)R(u)du} = \frac{-K[1 + \lambda(X)]R(X) + g(X)\bar{R}(X)}{v(X)\bar{R}(X)}$$

$$= \frac{g(X) - K\lambda(X) - K}{v(X)} \qquad (2.14)$$

If $m(u) = 0$, $k(u) = k$, and $\overline{h}_T(t) = 1$ (instantaneous repair), then $g(u) = k\lambda(u)$ and $v(u) = 1$. Upon substituting these expressions for $g(u)$ and $v(u)$ in (2.14), Beckmann's [4] result is recovered. Since

$$R(X) - R(0) = \int_0^X d[R(u)] = -\int_0^X [1 + \lambda(u)]R(u)du$$

from (2.5) and $R(0) = 1$, then

$$R(X) = 1 - \int_0^X [1 + \lambda(u)]R(u)du \qquad (2.15)$$

Substituting (2.15) in (2.14), the following is obtained

$$\frac{K + \int_0^X [g(u) - K\lambda(u) - Ki]\bar{R}(u)du}{\int_0^X v(u)\bar{R}(u)du} = \frac{g(X) - K\lambda(X) - Ki}{v(X)} \qquad (2.16)$$

Multiplying both sides of (2.16) by $v(X)\int_0^X v(u)\bar{R}(u)du$ yields

$$Kv(X) + v(X)\int_0^X [g(u) - K\lambda(u) - Ki]\bar{R}(u)du$$

$$= [g(X) - K\lambda(X) - Ki]\int_0^X v(u)\bar{R}(u)du$$

On rearranging the last expression, it follows that

$$\int_0^X v(u)[g(X) - K\lambda(X) - Ki]\bar{R}(u)du$$

$$- \int_0^X v(X)[g(u) - K\lambda(u) - Ki]\bar{R}(u)du = Kv(X)$$

Or

$$\int_0^X v(u)\left[\frac{g(X) - K\lambda(X) - Ki}{v(X)}\right]\bar{R}(u)du$$

$$- \int_0^X v(u)\left[\frac{g(u) - K\lambda(u) - Ki}{v(u)}\right]\bar{R}(u)du = K$$

Simplifying yields

$$\int_0^X v(u)\left[\frac{g(X) - K\lambda(X) - Ki}{v(X)} - \frac{g(u) - K\lambda(u) - Ki}{v(u)}\right]\bar{R}(u)du = K$$

which is equation (2.12). Now $\frac{g(u) - K\lambda(u) - Ki}{v(u)}$ is strictly increasing to infinity by assumption and $v(u)\bar{R}(u) \geq 0$ for $u \geq 0$. Thus, the integral

... left side of equation (2.12) is strictly increasing to infinity ... implies that a positive value for the equipment replacement age ... uniquely.

To show that X is the value for which $f_1(0)$ attains its absolute minimum, it is sufficient to establish that X is a relative minimum. Using (2.11), a condition for a relative minimum of $f_1(0)$ at X implies that $dx > 0$, the following inequality must hold

$$\frac{KR(X + dx) + \int_0^{X+dx} g(u)\hat{R}(u)du}{\int_0^{X+dx} v(u)\hat{R}(u)du} > \frac{KR(X) + \int_0^{X} g(u)\hat{R}(u)du}{\int_0^{X} v(u)\hat{R}(u)du} \tag{2.17}$$

Noting first that

$$\int_0^A v(u)\hat{R}(u)du = i\bar{h}_r(i)\int_0^A \hat{R}(u)du + [1 - \bar{h}_r(i)][1 - R(A)] > 0$$

for $A > 0$, inequality (2.17) can be written as

$$[KR(X + dx) + \int_0^{X+dx} g(u)\hat{R}(u)du]\int_0^X v(u)\hat{R}(u)du$$

$$> [KR(X) + \int_0^X g(u)\hat{R}(u)du]\int_0^{X+dx} v(u)\hat{R}(u)du$$

Multiplying and transposing yields

$$\left[\int_0^{X+dx} g(u)\hat{R}(u)du\right]\int_0^X v(u)\hat{R}(u)du - \left[\int_0^X g(u)\hat{R}(u)du\right]\int_0^{X+dx} v(u)\hat{R}(u)du$$

$$> KR(X)\int_0^{X+dx} v(u)\hat{R}(u)du - KR(X + dx)\int_0^X v(u)\hat{R}(u)du$$

Simplifying

$$\left[\int_X^{X+dx} g(u)\bar{R}(u)du\right] \text{[illegible]} - \left[\int_0^X g(u)\bar{R}(u)du\right] \int_X^{X+dx} v(u)\bar{R}(u)du$$

$$> [KR(X) - KR(X + dx)] \int_0^X v(u)\bar{R}(u)du - K\bar{R}(X) \int_X^{X+dx} v(u)\bar{R}(u)du \tag{2.18}$$

Rearranging (2.18) and using $\int_X^{X+dx} d[\bar{R}(u)] = \bar{R}(X + dx) - \bar{R}(X)$ yields

$$\left[\int_X^{X+dx} g(u)\bar{R}(u)du\right] \int_0^X v(u)\bar{R}(u)du$$

$$> - K\left[\int_X^{X+dx} d[\bar{R}(u)]\right] \int_0^X v(u)\bar{R}(u)du$$

$$+ \left[KR(X) + \int_0^X g(u)\bar{R}(u)du\right] \int_X^{X+dx} v(u)\bar{R}(u)du$$

Dividing both sides of the last expression by $\int_0^X v(u)\bar{R}(u)du$ results in the following inequality

$$\int_0^{X+dx} g(u)\bar{R}(u)du > - K \int_X^{X+dx} d[\bar{R}(u)]$$

$$+ \frac{KR(X) + \int_0^X g(u)\bar{R}(u)du}{\int_0^X v(u)\bar{R}(u)du} \left[\int_X^{X+dx} v(u)\bar{R}(u)du\right] \tag{2.19}$$

Using (2.5) and (2.14) in (2.19) yields

$$\int_X^{X+dx} g(u)X(u)du \quad K \int_X^{X+dx} [1 + \lambda(u)]\bar{R}(u)du$$

$$+ \frac{g(X) - X\lambda(X) - K\lambda}{v(X)} \left[\int_X^{X+dx} v(u)\bar{R}(u)du\right]$$

Transposing and collecting terms

$$\int_X^{X+dx} \left\{ [g(u) - K\lambda(u) - Ki] - v(u)\left[\frac{g(X) - K\lambda(X) - Ki}{v(X)}\right] \right\} \bar{R}(u)du > 0$$

Or

$$\int_X^{X+dx} v(u)\left[\frac{g(u) - K\lambda(u) - Ki}{v(u)} - \frac{g(X) - K\lambda(X) - Ki}{v(X)}\right] \bar{R}(u)du > 0 \qquad (2\text{-}20)$$

Since $\frac{g(u) - K\lambda(u) - Ki}{v(u)}$ is strictly increasing to infinity by assumption, then for all points $X \leq u \leq X + dx$ $(dx > 0)$,

$$\frac{g(u) - K\lambda(u) - Ki}{v(u)} > \frac{g(X) - K\lambda(X) - Ki}{v(X)}$$

and inequality (2-20) is satisfied. In a similar fashion it can be established that

$$\int_{X-dx}^{X} v(u)\left[\frac{g(X) - K\lambda(X) - Ki}{v(X)} - \frac{g(u) - K\lambda(u) - Ki}{v(u)}\right] \bar{R}(u)du > 0$$

The following corollaries to Theorem 2-1 are submitted without proof and are direct consequences of the theorem.

<u>Corollary 2-1</u>

If $\frac{g(u) - K\lambda(u)}{v(u)}$ $(u \geq 0)$ is strictly increasing to infinity and $\lambda(u)$ is strictly non-decreasing, then a unique finite optimal equipment replacement age X exists which is the unique solution to equation (2-12), and if the initial age, x, satisfies $x \leq X$, then an optimal X-policy exists for the major repair problem with arbitrary repair time distribution.

Corollary 2.2

If $\frac{g(u)}{v(u)}$ $(u \geq 0)$ is strictly increasing to infinity and [illegible] is strictly decreasing, then a unique finite optimal equipment replacement age X exists which is the unique solution to equation (2.12), and if the initial age, x, satisfies $x \leq X$, then an optimal X-policy exists for the major repair problem with arbitrary repair time distribution.

Corollary 2.3

If $[c(u) + k(u)\lambda(u)]$ and $r(u)$ are increasing functions to infinity and at least one of them is strictly increasing and if $\lambda(u) = \lambda$ = constant, then a unique finite optimal equipment replacement age X exists which is the unique solution to the equation

$$\int_0^X [g(X) - g(u)]\hat{R}(u)du = K$$

and if the initial age, x, satisfies $x \leq X$, then an optimal X-policy exists for the major repair problem with arbitrary repair time distribution.

Corollary 2.4

If $[c(u) + k(u)\lambda(u) - K\lambda(u)]$ $(u \geq 0)$ is strictly increasing to infinity, then a unique finite optimal equipment replacement age X exists which is the unique solution to the equation

$$\int_0^X \{[c(X) + k(X)\lambda(X) - K\lambda(X)] - [c(u) + k(u)\lambda(u) - K\lambda(u)]\}\hat{R}(u)du = K$$

and if the initial age, x, satisfies $x \leq X$, then an optimal X-policy exists for the major repair problem with instantaneous repair.

The function $\frac{g(u) - K\lambda(u) - Ki}{v(u)}$ is the sufficient

...on in Theorem 2.1 for the major repair problem with downtime ... similar to a corresponding function in the result obtained by Sivazlian [21] for the minimal repair problem with downtime. It is observed that Sivazlian's expression for the function $c(u)$ is equivalent to $\frac{g(u)}{v(u)}$ (see Section 1.3a and relations (2.2) and (2.9)). The difference in the functions is a multiple of the fixed cost of replacement K. The functional nature of this difference can be rationalized by recalling that major repair is equivalent to replacement with respect to the resulting equipment service age.

2.3 The Undiscounted Long Run Expected Cost Per Unit Time

In order to determine the undiscounted long run expected cost per unit time, denoted by $\overline{f}$, from results already obtained, the well-known result [4], [26]

$$\overline{f} = \lim_{i \to 0} [i f_1(U)]$$

is used. Utilizing (2.11) then

$$\overline{f} = \lim_{i \to 0} \left\{ \frac{iK\hat{R}(X) + i \int_0^X g(u)\hat{R}(u)du}{\int_0^X v(u)\hat{R}(u)du} \right\}$$

$$= \lim_{i \to 0} \left\{ \frac{K\hat{R}(X) + \int_0^X g(u)\hat{R}(u)du}{\int_0^X \frac{v(u)}{i}\hat{R}(u)du} \right\} \tag{2.21}$$

It is seen from (2.4) that

$$\lim_{i \to 0} \hat{R}(u) = R(u) \tag{2.22}$$

follows from (2.2)

$$\lim_{i \to 0} g(u) = \lim_{i \to 0} \left\{ c(u) + k(u)\lambda(u) + r(u)\left[\frac{1 - \overline{h}_T(i)}{i}\right]\lambda(u) \right\}$$

$$= c(u) + k(u)\lambda(u) + r(u)\lambda(u) \lim_{i \to 0} \left[\frac{1 - \overline{h}_T(i)}{i}\right]$$

Since $\lim_{i \to 0} \overline{h}_T(i) = 1$, l'Hospital's rule is employed. Thus

$$\lim_{i \to 0} g(u) = c(u) + k(u)\lambda(u) - r(u)\lambda(u) \lim_{i \to 0} \overline{h}_T'(i)$$

Providing $E(T)$ exists, $E(T) = - \lim_{i \to 0} \overline{h}_T'(i)$ and the following is obtained

$$\lim_{i \to 0} g(u) = c(u) + k(u)\lambda(u) + r(u)\lambda(u)E(T) \tag{2.23}$$

Using (2.9)

$$\lim_{i \to 0} \frac{v(u)}{i} = \lim_{i \to 0} \left\{ 1 + \lambda(u)\left[\frac{1 - \overline{h}_T(i)}{i}\right] \right\}$$

$$= 1 + \lambda(u) \lim_{i \to 0} \left[\frac{1 - \overline{h}_T(i)}{i}\right]$$

$$= 1 + \lambda(u)E(T) \tag{2.24}$$

Substituting appropriately (2.22), (2.23), and (2.24) in (2.21) yields

$$\overline{T} = \frac{KR(K) + \int_0^K [c(u) + k(u)\lambda(u) + r(u)\lambda(u)E(T)]R(u)du}{\int_0^K [1 + \lambda(u)E(T)]R(u)du} \tag{2.25}$$

[illegible] [1] result for the major repair problem with zero [illegible] is recovered if $c(u) = 0$, $k(u) = k$ = constant, and $E(T) = 0$ (zero downtime). In addition, from Corollary 2.4, the existence and uniqueness of an optimal solution is assured provided that $k > K$ and $\lambda(u)$ is strictly increasing to infinity. Schaeffer's [20] result is recovered by setting $c(u) = su^{\theta}$, $k(u) = k$ = constant, and $E(T) = 0$

CHAPTER 3

AGE REPLACEMENT WITH PARTIAL REPAIR AND DOWNTIME

3.1 Introduction

This chapter examines the hysteresis effect when an age replacement policy, called an X-policy, is being followed. In particular, the linear hysteresis problem with non-zero downtime for partial repair and with constant failure rate is investigated. For the functional-differential equation (1.10) with failure rate function, $\lambda(x)$, equal to λ, and with hysteresis age function, $a(x)$, equal to ax $(0 \leq a \leq 1)$, a general solution is determined. This is the sum of two solutions, namely, a complementary and a particular solution.

The complementary solution is determined by assuming it is of the series form. As a consequence, the function $S(x,\theta,\lambda\bar{h}_T(1),a)$ is obtained, and is examined analytically and graphically in Appendix B. The properties of $S(x,\theta,\lambda\bar{h}_T(1),a)$ and its derivatives with respect to x are important in obtaining the results in this and the following chapter. Due to the structure of the cost function $g(x)$, when $\lambda(x)$ equals λ, it is assumed that the cost function is a member of the class of nth order polynomials, whose coefficients are all positive, in order to obtain the particular solution. The assumption that all the coefficients of the polynomial are positive is later established as one of a pair of sufficient conditions for a unique finite optimal equipment replacement age x to exist.

Three particular polynomial forms with positive coefficients are examined in this chapter. They are: constant, linear, and quadratic

for linear and quadratic polynomials, a pair of sufficient conditions is determined so that a unique optimal equipment replacement age X exists for the linear hysteresis problem, whereas, for constant polynomials, only one sufficient condition is determined Numerical examples illustrating the linear and quadratic polynomial forms are discussed.

3.2 General Solution to Functional-Differential Equation with $\lambda(s) = \lambda$ and $a(x) = ax$

The functional-differential equation (1.10) of Chapter 1 is recalled. It is

$$f_1'(x) - [i + \lambda(x)]f_1(x) + \lambda(x)\bar{h}_T(i)f_1[a(x)] = -g(x) \qquad (3.1)$$

where

$$g(x) = c(x) + k(x)\lambda(x) + \frac{r(x)}{i}[1 - \bar{h}_T(i)]\lambda(x) \qquad (3.2)$$

If the hysteresis age function, $a(x)$, is equal to ax (where $0 \leq a \leq 1$) in (3.1), the resulting linear differential-difference equation has variable coefficients due to the failure rate function $\lambda(x)$ As yet, no solution has been found by the author to this differential equation However, it is possible to obtain a solution by setting the failure rate function, $\lambda(x)$, equal to λ, a constant letting $a(s) = ax$ (where $0 \leq a \leq 1$) in (3.1) and $\lambda(x) = \lambda$ = constant in (3.1) and (3.2) yields

$$f_1'(x) - (i + \lambda)f_1(x) + \lambda\bar{h}_T(i)f_1(ax) = -\bar{g}(x) \qquad (3.3)$$

where

$$\bar{g}(x) = c(x) + \lambda k(x) + \frac{\lambda r(x)}{i}[1 - \bar{h}_T(i)] \qquad (3.4)$$

[illegible] the general solution to (3 3) consists of [illegible] ary solution, a particular solution, and the constant ter[illegible] solutions by evaluating the general solution at $x = 0$.

It is easily verified (Appendix A) that the general solution to (3 3) is of the form

$$f_1(x) = y_1(x) + y_2(x) \tag{3 5}$$

where $y_1(x)$ is a solution to the homogeneous equation

$$y_1'(x) - (1 + \lambda)y_1(x) + \lambda\bar{h}_T(t)y_1(ax) = 0 \tag{3 6}$$

and $y_2(x)$ is a solution to the non-homogeneous equation

$$y_2'(x) - (1 + \lambda)y_2(x) + \lambda\bar{h}_T(t)y_2(ax) = - g(x) \tag{3 7}$$

Equation (3 6) can be solved by assuming a series solution of the form

$$y_1(x) = \sum_{j=0}^{\infty} a_j x^j \tag{3 8}$$

where a_j, $j=0,1,2,\ldots$, are constants to be determined.

Using relation (3 8) in (3 6), the following identity in x^j, $j=0,1,2,\ldots$, is obtained

$$\sum_{j=0}^{\infty} (j+1)a_{j+1}x^j - (1 + \lambda) \sum_{j=0}^{\infty} a_j x^j + \lambda\bar{h}_T(t) \sum_{j=0}^{\infty} a_j a^j x^j = 0$$

Collecting terms in x^j in the last expression yields

$$\sum_{j=0}^{\infty} [(j+1)a_{j+1} - (1 + \lambda)a_j + \lambda\bar{h}_T(t)a_j a^j]x^j = 0$$

h, providing $x \neq 0$, results in the difference equation

$$(j+1)a_{j+1} - (1+\lambda)a_j + \lambda\bar{h}_T(1)a_j s^j = 0 \quad , \quad j=0,1,2,.. \tag{3.9}$$

The solution to (3.9) obtained recursively is

$$a_j = a_0 \frac{1}{j!} \prod_{k=0}^{j-1} (1 + \lambda - \lambda\bar{h}_T(1)s^k) \quad , \quad j \geq 1$$

where a_0 is an arbitrary constant whose value is to be determined. From (3.8) then

$$y_1(x) = a_0\left[1 + \sum_{j=1}^{\infty} \frac{x^j}{j!} \prod_{k=0}^{j-1} (1 + \lambda - \lambda\bar{h}_T(1)s^k)\right] \tag{3.10}$$

Define $\Theta = 1 + \lambda$ and

$$S(x,\Theta,\lambda\bar{h}_T(1),s) = 1 + \sum_{j=1}^{\infty} \frac{x^j}{j!} \prod_{k=0}^{j-1} (\Theta - \lambda\bar{h}_T(1)s^k) \tag{3.11}$$

The properties of the function $S(x,\Theta,\lambda\bar{h}_T(1)s)$ are discussed in detail in Appendix 8. The pertinent ones for this chapter are

(i) for all non-negative real values of x, $S(x,\Theta,\lambda\bar{h}_T(1),s)$ is a convergent power series;

(ii) all derivatives of $S(x,\Theta,\lambda\bar{h}_T(1),s)$ with respect to x exist and are positive for all non-negative real values of x, and

(iii) $[S(x,\Theta,\lambda\bar{h}_T(1),s) - 1]$ is positive for all positive real values of x.

Equation (3.10) can be rewritten using (3.11) as

$$y_1(x) = a_0 S(x,\Theta,\lambda\bar{h}_T(1),s) \tag{3.12}$$

When $a = 1$, as in the minimal repair problem,

$$S(x,\theta,\lambda\bar{h}_T(1),1) = e^{(\theta - \lambda\bar{h}_T(1))x}$$

and

$$y_1(x) = c_0 e^{(\theta - \lambda\bar{h}_T(1))x}$$

Similarly when $a = 0$, as in the major repair problem,

$$S(x,\theta,\lambda\bar{h}_T(1),0) = (\theta - \lambda\bar{h}_T(1))\,\frac{(e^{\theta x} - 1)}{\theta} + 1$$

and

$$y_1(x) = c_0(\theta - \lambda\bar{h}_T(1))\,\frac{(e^{\theta x} - 1)}{\theta} + c_0$$

For notational convenience, let

$$S(x,a) = S(x,\theta,\lambda\bar{h}_T(1),a) \tag{3.13}$$

where

$$S(x,a) = 1 + \sum_{j=1}^{\infty} \frac{x^j}{j!} \prod_{k=0}^{j-1} F_k(a) \tag{3.14}$$

and

$$F_k(a) = \theta - \lambda\bar{h}_T(1)a^k \quad , \qquad k=0,1,2,\cdots \tag{3.15}$$

where $a^0 = 1$, for all a. Then substituting (3.13) into (3.12) yields

$$y_1(x) = c_0 S(x,a) \tag{3.16}$$

To determine a particular solution $y_2(x)$, it is necessary to solve the linear differential-difference equation (3.7). It is observed that the term $y_2(ax)$ in equation (3.7) constitutes a backward differencing, since $0 \leq a \leq 1$. The effect of this term in equation (3.7) requires an assumption about the nature of $\bar{g}(x)$ in order to

[illegible] is defined by the relation (3 4). Since $c(x)$, $r(x)$, and $r(x)$ are unspecified, it is assumed that each can be represented by a polynomial. Then their linear combination defined by (3.4) is also a polynomial. Assume that $\hat{g}(x)$ is a member of the class of nth order polynomials whose coefficients are all required to be positive. Then $g(x)$ is polynomial of the form

$$g(x) = a_n x^n + a_{n-1} x^{n-1} + \quad + a_1 x + a_0 \tag{3 17}$$

where

$$a_i > 0 \ , \quad i=0,1, \quad ,n$$

The assumption $a_1 > 0$ for $1 \geq 1$ is later established as one of a pair sufficient conditions for a unique finite optimal equipment replacement age x to exist. Since there is no physical interpretation for a negative cost in the repair and replacement problems under investigation, it is assumed that $c_0 > 0$. Since $\hat{g}(x)$ has all positive coefficients, then $\hat{g}(x)$ is a convex function. The particular solution $y_2(x)$ to (3 7) with $g(x)$ given by (3.17) is

$$y_2(x) = b_n x^n + b_{n-1} x^{n-1} + \quad + b_1 x + b_0 \tag{3 18}$$

where the terms b_j are determined by the solution to the difference equation

$$b_{j-1} - \frac{j}{P_{j-1}(a)} b_j = \frac{a_{j-1}}{P_{j-1}(a)} \ , \quad j=1, \quad ,n$$

with

$$b_n = \frac{c_n}{P_n(a)}$$

Solving (3.19) recursively yields

$$b_j = \frac{a_j}{F_j(s)} + \frac{1}{\prod_{k=j}^{n} F_k(s)} \left[\sum_{i=j+1}^{n-1} (j+1) \cdots (i) a_i \prod_{k=i+1}^{n} F_k(s) + (j+1) \cdots (n) a_n \right], \quad j=0, \ldots, n-2$$

$$b_{n-1} = \frac{a_{n-1}}{F_{n-1}(s)} + \frac{n a_n}{F_{n-1}(s) F_n(s)} \tag{3 20}$$

and

$$b_n = \frac{a_n}{F_n(s)}$$

Using (3 20) in (3 18) yields the particular solution

$$y_2(x) = \frac{a_n x^n}{F_n(s)} + \frac{[a_{n-1} F_{n-1}(s) + n a_n] x^{n-1}}{F_{n-1}(s) F_n(s)} + \sum_{j=0}^{n-2} x^j \left\{ \frac{a_j}{F_j(s)} + \frac{1}{\prod_{k=j}^{n} F_k(s)} \left[\sum_{i=j+1}^{n-1} (j+1) \cdots (i) a_i \prod_{k=i+1}^{n} F_k(s) + (j+1) \cdots (n) a_n \right] \right\} \tag{3 21}$$

which is a solution to (3 7) with the nth order polynomial g(x) given by (3 17).

Thus, the general solution to equation (3 3) with $\dot{g}(x)$ given by (3 17) is

$$f_1(x) = y_1(x) + y_2(x)$$

$$= a_0 S(x,s) + \frac{a_n x^n}{F_n(s)} + \frac{[a_{n-1} F_{n-1}(s) + n a_n] x^{n-1}}{F_{n-1}(s) F_n(s)} + \sum_{j=0}^{n-2} x^j \frac{a_j}{F_j(s)} + \frac{1}{\prod_{k=j}^{n} F_k(s)} \left[\sum_{i=j+1}^{n-1} (j+1) \cdots (i) a_i \prod_{k=i+1}^{n} F_k(s) + (j+1) \cdots (n) a_n \right] \tag{3 22}$$

It is necessary to evaluate the arbitrary constant a_0 in (3.22).
For $x = 0$ in (3.14)

$$S(0,s) = 1$$

and evaluating (3.22) at $x = 0$ yields

$$a_0 = f_1(0) - \frac{1}{\prod_{k=0}^{n} P_k(s)} \left[\sum_{i=0}^{n-1} (i!)a_i \prod_{k=i+1}^{n} P_k(s) + (n!)a_n \right]$$

Substituting the expression just determined for a_0 in (3.22), the following is obtained

$$f_1(x) = f_1(0)S(x,s) + \frac{c_n x^n}{P_n(s)} + \frac{[a_{n-1}P_{n-1}(s) + na_n]x^{n-1}}{P_{n-1}(s)P_n(s)}$$

$$+ \sum_{j=0}^{n-2} x^j \left\{ \frac{a_j}{P_j(s)} + \frac{1}{\prod_{k=j}^{n} P_k(s)} \left[\sum_{i=j+1}^{n-1} (j+1) \cdots (i)a_i \prod_{k=i+1}^{n} P_k(s) \right. \right.$$

$$\left. \left. + (j+1) \cdots (n)c_n \right] \right\} - \frac{1}{\prod_{k=0}^{n} P_k(s)} \left[\sum_{i=0}^{n-1} (i!)a_i \prod_{k=i+1}^{n} P_k(s) \right.$$

$$\left. + (n!)a_n \right] [S(x,s) - 1] \qquad (3.23)$$

Assuming that $g(s)$ is an nth order polynomial, this last expression is the general solution to the functional-differential equation (3.3). The following section applies (3.23) in the discussion of three particular polynomial forms $g(x)$.

3.3 Special Polynomial Forms of $g(x)$

Three polynomial forms of $\hat{g}(x)$ are investigated in this section, namely, constant, linear, and quadratic, where every coefficient of each polynomial $g(x)$ is assumed to be positive.

a. $g(x) = c_0$

When $\hat{g}(x)$ is a constant, this implies that one of three alternatives exists. Either the instantaneous repair cost, the operations cost, and the loss of revenue per unit of downtime cost are all non-zero constants, or two of the costs are non-zero constants and one is zero, or one of the costs is a non-zero constant and two are zero. Under each of these three possibilities there is never an incentive to replace, which implies that the optimal replacement age is infinity. This conjecture will be established.

Using (3.3) with $\hat{g}(a) = c_0$ yields the functional-differential equation

$$f_1'(x) - (i + \lambda)f_1(x) + \lambda\bar{h}_T(i)f_1(ax) = -c_0 \qquad (3.24)$$

Then applying (3.23), which is the general solution to (3.3), to equation (3.24), the general solution to (3.24) can be written as

$$f_1(x) = f_1(0)S(x,a) - \frac{c_0}{P_0(a)}\,[S(x,a) - 1] \qquad (3.25)$$

If an optimal X-policy is followed, continuity of the function $f_1(x)$ at $x = X$ is required. It then follows that

$$f_1(X) = K(X) + f_1(0) \qquad (3.26)$$

Applying (3.26) in (3.25), it is required that at $x = X$

$$K(X) + f_1(0) = f_1(0)S(X,a) - \frac{c_0}{F_0(a)}\,[S(X,a) - 1]$$

Solving for $f_1(0)$ yields

$$f_1(0) = \frac{K(X)}{S(X,a) - 1} + \frac{c_0}{F_0(a)} \tag{3 27}$$

Since $K(X)$ is continuous in X, and further it is a bounded function, whereas $S(X,a)$ is a strictly increasing function in X to infinity, thus $f_1(0)$ is minimized with respect to X as X approaches infinity. The following theorem summarizes the result

Theorem 3 1

A sufficient condition for a unique optimal equipment replacement age X to exist for the linear hysteresis problem where $a(x) = ax$, $h(t) = \lambda$ = constant, and $g(x) = c_0$ is that X be infinite

b. $g(x) = c_1x + c_0$

Using (3 3) with $g(x) = c_1x + c_0$ yields the functional-differential equation

$$f_1'(x) - (1 + \lambda)f_1(x) + \lambda\bar{h}_2(t)f_1(ax) = -c_1x - c_0 \tag{3 28}$$

Applying (3 23), which is the general solution to (3 3), to (3 28), the following is obtained

$$f_1(x) = f_1(0)S(x,a) + \frac{c_1x}{F_1(a)} - \frac{c_1 + c_0F_1(a)}{F_0(a)F_1(a)}\,[S(x,a) - 1] \tag{3 29}$$

Using (3 26) in (3 29), it follows that at $x = X$

$$K(X) + f_1(0) = f_1(0)S(X,a) + \frac{c_1X}{F_1(a)} - \frac{c_1 + c_0F_1(a)}{F_0(a)F_1(a)}\,[S(X,a) - 1]$$

Collecting terms in $f_1(0)$ and solving for $f_1(0)$ yields

$$f_1(0) = \frac{K(X) - \dfrac{c_1 X}{F_1(s)}}{S(X,s) - 1} + \frac{c_1 + c_0 F_1(s)}{F_0(s) F_1(s)}$$

For the remainder of the discussion of the linear form of $g(x)$ the analysis is restricted to the situation where the fixed cost of replacement $K(X)$ is independent of the replacement age X. Since $K(X) = K =$ constant, then

$$f_1(0) = \frac{K - \dfrac{c_1 X}{F_1(s)}}{S(X,s) - 1} + \frac{c_1 + c_0 F_1(s)}{F_0(s) F_1(s)} \tag{3.30}$$

The following theorem presents the results obtained for the linear form of $g(x)$

<u>Theorem 3.2</u>

Sufficient conditions for a unique finite optimal equipment replacement age X to exist for the linear hysteresis problem where $a(x) = ax$, $\lambda(x) = \lambda =$ constant, $K(X) = K =$ constant, and $g(x) = u_1 x + c_0$ are that $c_1 > 0$ and

$$\frac{u_1}{F_1(s)} [S(X,s) - \lambda] + \left[K - \frac{c_1 X}{F_1(s)} \right] S_X(X,s) = 0 \tag{3.31}$$

where X is the unique solution to (3.31)

<u>Proof</u>

If the initial equipment age x is such that $x < X$, then it is clear from substituting (3.30) into (3.29) that if an optimal replacement age X exists, it minimizes both $f_1(0)$ and $f_1(x)$ for all x

Differentiating the expression for $f_1(0)$ in (3.30) with respect to X and equating the result to zero, a sufficient condition for the existence of an optimal equipment replacement age X, say X^*, is obtained and is given by the equation in X^*

$$\frac{c_1}{F_1(s)}\,[S(X^*,s) - 1] + \left[K - \frac{c_1 X^*}{F_1(s)}\right] S_X(X^*,s) = 0 \qquad (3.32)$$

where

$$S_X(X,s) = \frac{dS(X,s)}{dX}$$

Note that (3.32) is identical to (3.31). Dividing all terms in (3.30) by $\dfrac{c_1 S_X(X^*,s)}{F_1(s)}$ and rearranging yields

$$X^* - \frac{S(X^*,s) - 1}{S_X(X^*,s)} = \frac{KF_1(s)}{c_1} \qquad (3.33)$$

Taking the second derivative of $f_1(0)$ with respect to X results in

$$\frac{d^2 f_1(0)}{dX^2} = \frac{\left(-\dfrac{c_1 S_X(X,s)}{F_1(s)} + \dfrac{c_1 S_X(X,s)}{F_1(s)} - [K - \dfrac{c_1 X}{F_1(s)}]S_{XX}(X,s)\right)[S(X,s) - 1]^2}{[S(X,s) - 1]^4}$$

$$- \frac{2S_X(X,s)[S(X,s) - 1]\left(-\dfrac{c_1}{F_1(s)}[S(X,s)-1] - [K - \dfrac{c_1 X}{F_1(s)}]S_X(X,s)\right)}{[S(X,s) - 1]^4}$$

(3.34)

where

$$S_{XX}(X,s) = \frac{d^2 S(X,s)}{dX^2}$$

Substituting (3.32) into (3.34) and simplifying

$$\frac{d^2 f_1(0)}{dX^2}\Bigg|_{X=X^*} = \text{[illegible]} \qquad (3\ 33)$$

Recall that for all positive values of X, the functions $S(X,e) - 1$, $S_X(X,e)$, and $S_{XX}(X,e)$ are positive. Returning to equation (3 33), let L(X) equal the left-hand side of the equation that is

$$L(X) = X - \frac{S(X,e) - 1}{S_X(X,e)}$$

Differentiating L(X) with respect to X yields

$$\frac{dL(X)}{dX} = 1 - \frac{S_X(X,e)S_X(X,e) - S_{XX}(X,e)[S(X,e) - 1]}{[S_X(X,e)]^2}$$

$$= \frac{S_{XX}(X,e)[S(X,e) - 1]}{[S_X(X,e)]^2} > 0 \qquad \text{for all } x > 0$$

Furthermore, it is easily verified that $L(0) = 0$ Therefore, the function $L(X) = X - [S(X,e) - 1]/S_X(X,e)$ is positive for $X > 0$ and is strictly increasing to infinity. If $c_1 > 0$ then $XP_1(e)/c_1 > 0$ It therefore follows from (3 33) that the optimum equipment replacement age X, if it exists, is unique.

The existence of an optimal equipment replacement age X which minimizes $f_1(0)$ is established by showing that $\left.\frac{d^2 f_1(0)}{dX^2}\right|_{X=X^*} > 0$.

Equation (3 32) can be rearranged to yield

$$X - \frac{c_1 X^*}{P_1(e)} = - \frac{c_1[S(X^*,e) - 1]}{P_1(e)S_X(X^*,e)} \qquad (3\ 36)$$

Using (3.34) in (3.35) and if $c_1 > 0$, the following is obtained

$$\left.\frac{d^2 F_1(0)}{dX^2}\right|_{X=X^*} = \frac{c_1 S_{XX}(X^*,a)}{F_1(a) S_X(X^*,a)[S(X^*,a) - 1]} > 0$$

which proves existence. Then the condition $c_1 > 0$ is one of a pair of sufficient conditions for a unique finite optimal equipment replacement age X to exist.

The following is a numerical example of the linear hysteresis problem with $g(x) = c_1 e + c_0$.

Example 1

Consider a continuously operating equipment subject to a constant failure rate and with an expected number of failures of 10 per year. Following each failure, the equipment is instantly repaired. Repair of an equipment results in a recovery of usable service life, where the amount of recovery is proportional to the age of the equipment. Assume that a repair recovers half of the service life used by the equipment. The combined repair and operating cost in \$ per year of service for the equipment when service age is u is 10.04 u + 7.2. It is anticipated that when the existing equipment is in need of being replaced, the new equipment will have identical characteristics and a fixed cost of \$350 will be incurred. Assuming a fixed interest rate of 10% per year, it is required to determine the service life at which it is more economical to replace the existing equipment rather than to continue operating and repairing it.

From the problem statement, it is known that K = \$350,

[illegible] /year, $a = 1/$ [illegible] h_1(…) = …1/year, $F_1(1$ [illegible] $c_1 = 10.04$, and $c_0 = 7.2$. Then solving numerically equation (3.31), it is determined that the most economic life at which an equipment should be replaced is 10 years.

The linear cost function $g(x) = 10x$ is examined in Tables 3.1 through 3.5 for specific values of a. These values are 0, 0.25, 0.50, 0.75, and 1. Each table presents the optimum economic life in years and the corresponding value of $f_1(0)$ for various combinations of K, θ, and $\lambda\bar{h}_T(i)$.

From Tables 3.1 through 3.5, the following observations can be made concerning the overall behavior of X and $f_1(0)$ as the variables K, θ, $\lambda\bar{h}_T(i)$, and a are varied one at a time. As the replacement cost K increases, both X and $f_1(0)$ increase. This is logical since increasing the replacement cost would delay replacements and incur (in addition to the increase in K) extra operating and repair costs due to the delay in replacing. As θ increases, the replacement age X increases and $f_1(0)$ decreases. Since $\theta = i + \lambda$, an increase in θ while $\lambda\bar{h}_T(i)$ is fixed corresponds to an increase in the interest rate i. Thus future costs are additionally discounted and their is an economic incentive to retain an equipment longer. As $\lambda\bar{h}_T(i)$ increases, the replacement age X decreases and $f_1(0)$ increases. Due to the structure of $S(X,a)$, an exception to the last statement occurs when $a = 0$, in which case X does not vary and $f_1(0)$ increases as $\lambda\bar{h}_T(i)$ increases. With θ fixed, an increase in $\lambda\bar{h}_T(i)$ corresponds to an increase in the failure rate λ. Since an equipment fails more often, there are additional repair costs incurred and the equipment is replaced sooner. As a increases from 0 to 1, the replacement age X decreases and $f_1(0)$ increases. Increasing a corresponds to less service life being recovered at the termination of a repair. Thus higher operating and repair costs are incurred and the equipment is replaced more often. For the case of minimal repair, that is $a = 1$, the values obtained

Table 3.1 Table of Optimum X and Corresponding $f_1(0)$

$g(x) = 10x$ When $a = 0.0$

		X							
		50		150		250		350	
b	$\lambda h_T(t)$	X	$f_1(0)$	X	$f_1(0)$	X	$f_1(0)$	X	$f_1(0)$
0.20	0.10	3.54	253.4	6.69	368.7	9.21	420.7	11.50	449.9
0.40	0.10	3.99	66.5	8.41	80.5	12.48	82.8	16.50	83.2
	0.20	3.99	99.8	8.41	120.7	12.48	124.2	16.50	124.9
	0.30	3.99	199.4	8.41	241.4	12.48	248.3	16.50	249.7
0.60	0.10	4.56	31.2	10.66	33.3	16.67	33.3	22.67	33.3
	0.20	4.56	39.0	10.66	41.6	16.67	41.7	22.67	41.7
	0.30	4.56	52.0	10.66	55.5	16.67	55.5	22.67	55.6
	0.40	4.56	77.9	10.66	83.2	16.67	83.3	22.67	83.3
	0.50	4.56	155.9	10.66	166.4	16.67	166.7	22.67	166.7
0.80	0.10	5.23	17.6	13.25	17.9	21.25	17.9	29.25	17.9
	0.20	5.23	20.5	13.25	20.8	21.25	20.8	29.25	20.9
	0.30	5.23	24.6	13.25	25.0	21.25	25.0	29.25	25.0
	0.40	5.23	30.8	13.25	31.2	21.25	31.3	29.25	31.3
	0.50	5.23	41.5	13.25	41.7	21.25	41.7	29.25	41.7
	0.60	5.23	61.5	13.25	62.5	21.25	62.5	29.25	62.5
	0.70	5.23	123.1	13.25	125.0	21.25	125.0	29.25	125.0

Table [illegible] Table of Optimum X and Corresponding $f_1(0)$ [illegible]
$g(x) = 10x$ When [illegible]

		K							
		50		150		250		350	
θ	$\lambda\bar{h}_T(1)$	X	$f_1(0)$	X	$f_1(0)$	X	$f_1(0)$	X	$f_1(0)$
0.20	0.10	3.43	262.9	6.35	392.9	8.63	456.0	10.68	493.8
0.40	0.10	3.88	68.9	8.05	85.0	11.87	88.1	15.63	88.7
	0.20	3.77	107.2	7.69	133.1	11.25	141.0	14.75	142.4
	0.30	3.65	222.6	7.33	286.9	10.64	302.0	13.89	306.1
0.60	0.10	4.44	32.2	10.29	34.7	16.04	34.8	21.79	34.8
	0.20	4.32	41.7	9.92	45.3	15.42	45.5	20.92	45.5
	0.30	4.20	57.6	9.54	63.2	14.79	63.5	20.04	63.5
	0.40	4.09	89.5	9.17	99.5	14.17	100.0	19.17	100.0
	0.50	3.97	185.5	8.80	209.1	13.54	210.4	18.29	210.5
0.80	0.10	5.11	18.1	12.88	18.4	20.63	18.4	28.38	18.4
	0.20	4.98	21.8	12.50	22.2	20.00	22.2	27.50	22.2
	0.30	4.86	26.9	12.13	27.6	19.38	27.6	26.63	27.6
	0.40	4.74	34.8	11.75	35.7	18.75	35.7	25.75	35.7
	0.50	4.62	47.9	11.38	49.4	18.13	49.4	24.88	49.4
	0.60	4.50	74.2	11.00	76.9	17.50	76.9	24.00	76.9
	0.70	4.38	153.5	10.63	160.0	16.88	160.0	23.13	160.0

Table 3.3 Table of [illegible]

$g(x) =$ [illegible]

α	$\bar{b}_2(s)$	50		150		250		350	
		X	$f_1(0)$	X	$f_1(0)$	X	$f_1(0)$	X	$f_1(0)$
0.20	0.10	3.35	271.5	6.10	418.3	6.19	491.9	10.02	540.9
0.40	0.10	3.79	71.3	7.73	89.9	11.26	93.8	14.78	94.9
	0.20	3.60	114.7	7.07	151.8	10.11	162.0	13.06	185.2
	0.30	3.41	248.3	6.44	243.5	8.69	377.4	11.44	391.0
0.60	0.10	4.34	32.3	9.93	36.3	15.42	36.4	20.92	[illegible]
	0.20	4.12	44.6	9.20	49.7	14.17	50.0	19.17	[illegible]
	0.30	3.91	62.8	8.47	73.3	12.94	74.0	17.42	[illegible]
	0.40	3.71	102.7	7.77	122.7	11.70	124.6	15.69	[illegible]
	0.50	3.51	220.9	7.08	276.0	10.49	284.4	13.94	[illegible]
0.80	0.10	4.99	18.7	12.50	19.0	20.00	19.0	27.50	19.0
	0.20	4.76	23.1	11.75	22.6	18.75	22.8	25.75	[illegible]
	0.30	4.52	29.8	11.01	30.6	17.56	30.8	24.00	[illegible]
	0.40	4.30	39.5	10.28	41.6	16.25	41.7	22.25	[illegible]
	0.50	4.06	56.5	9.52	60.5	15.00	60.6	20.50	[illegible]
	0.60	3.87	91.1	8.76	99.6	13.75	100.0	18.75	[illegible]
	0.70	3.66	196.0	8.05	221.1	12.51	222.2	17.00	[illegible]

Table 3.4 Table of Optimum X and Corresponding $f_1(0)$ for $g(x) = 10x$ When $a = 0.75$

		h							
		50		150		250		350	
s	$\lambda \bar{b}_2(1)$	X	$f_1(0)$	X	$f_1(0)$	X	$f_1(0)$	X	$f_1(0)$
0.20	0.10	3.32	278.6	5.98	436.9	7.96	525.1	9.65	585.2
0.40	0.10	3.74	73.4	7.50	94.9	10.81	100.3	14.04	101.9
	0.20	3.51	121.4	6.87	169.2	9.28	185.6	11.74	194.2
	0.30	3.29	267.8	5.94	404.1	7.94	470.4	9.72	509.4
0.60	0.10	4.27	36.5	9.61	37.9	14.82	38.1	20.06	38.1
	0.20	3.99	47.4	8.59	54.9	13.01	55.5	17.46	55.6
	0.30	3.73	69.9	7.63	86.0	11.25	88.5	14.91	88.9
	0.40	3.49	116.0	6.74	153.8	9.60	163.4	12.44	165.9
	0.50	3.27	256.6	5.96	369.6	8.11	413.5	10.16	432.4
0.80	0.10	4.90	19.2	12.14	19.7	19.38	19.7	26.63	19
	0.20	4.59	24.6	11.04	25.6	17.51	25.6	24.00	25.6
	0.30	4.29	32.5	9.95	34.7	15.65	34.8	21.38	34.8
	0.40	4.00	45.0	8.89	49.8	13.80	50.0	18.77	50.0
	0.50	3.74	66.4	7.89	77.6	11.97	78.4	16.16	78.4
	0.60	3.49	110.5	6.92	138.4	10.20	142.4	13.59	142.8
	0.70	3.26	245.1	6.07	334.0	8.54	357.2	11.09	362.5

Table 3.5 Table of Optimum X and Cor[illegible]

$g(x) = 10x$ When $a =$ [illegible]

		K							
		50		150		250		350	
e	$\lambda \bar{h}_T(t)$	X	$f_1(0)$	X	$f_1(0)$	X	$f_1(0)$	X	$f_1(0)$
0.20	0.10	3.34	283.8	6.03	452.6	8.01	551.2	9.71	621.5
0.40	0.10	3.75	75.1	7.48	99.3	10.70	106.6	13.78	109.3
	0.20	3.54	126.7	6.68	184.4	9.21	210.4	11.50	224.9
	0.30	3.34	283.8	6.03	452.6	8.01	551.2	9.71	621.5
0.60	0.10	4.26	35.3	9.48	39.7	14.50	40.0	19.58	40.0
	0.20	3.99	49.9	8.41	60.3	12.48	62.1	16.50	62.4
	0.30	3.75	75.1	7.48	99.3	10.70	106.6	13.78	109.3
	0.40	3.54	126.7	6.69	184.4	9.21	210.4	11.50	224.8
	0.50	3.34	283.8	6.03	452.6	8.01	551.2	9.71	621.5
0.80	0.10	4.88	19.7	11.83	20.4	18.93	20.4	25.93	20.4
	0.20	4.56	26.0	10.66	27.7	16.67	27.8	22.67	27.8
	0.30	4.25	35.3	9.48	39.7	14.50	40.0	19.50	40.0
	0.40	3.99	49.9	8.41	60.3	12.48	62.1	16.50	62.4
	0.50	3.75	75.1	7.48	99.3	10.70	106.6	13.78	109.3
	0.60	3.54	126.7	6.69	184.4	9.21	210.4	11.50	224.9
	0.70	3.34	283.8	6.03	452.6	8.01	551.2	9.71	621.5

[illegible] $f_1(0)$ for a particular X are a function of $\theta - \lambda\bar{h}_T(1)$, due [illegible] structure of $S(X,a)$. Thus, the value of X and $f_1(0)$ for $\theta = 0.4$ [illegible] $) = 0.2$ are the same as for $\theta = 0.6$ and $\lambda\bar{h}_T(1) = 0.4$

[illegible] evaluation can be made concerning the X-policy for various values [illegible] a for the linear hysteresis problem from the results displayed in the tables. This is due to the cost function $g(x)$ being the same for each value of a, and therefore a minimal repair at age x costs the same as a major repair. Chapter 4 addresses itself to the problem of determining an optimal a and X.

c. $\underline{g(x) = c_2x^2 + c_1x + c_0}$

Letting $g(a) = c_2x^2 + c_1x + c_0$ in (3.3) results in the functional differential equation

$$f_1'(x) - (1 + \lambda)f_1(x) + \lambda\bar{h}_T(t)f_1(ax) = -c_2x^2 - c_1x - c_0 \qquad (3.37)$$

Using (3.29) which is the general solution to (3.3), the general solution to (3.37) can be written as

$$f_1(x) = f_1(0)S(x,a) + \frac{c_2x^2}{P_2(a)} + \frac{[2c_2 + c_1P_2(a)]x}{P_1(a)P_2(a)}$$

$$- \frac{2c_2 + c_1P_2(a) + c_0P_1(a)P_2(a)}{P_0(a)P_1(a)P_2(a)}[S(x,a) - 1] \qquad (3.38)$$

Applying (3.26) in (3.38), it follows that at $x = X$

$$K(X) + f_1(0) = f_1(0)S(X,a) + \frac{c_2X^2}{P_2(a)} + \frac{[2c_2 + c_1P_2(a)]X}{P_1(a)P_2(a)}$$

$$- \frac{2c_2 + c_1P_2(a) + c_0P_1(a)P_2(a)}{P_0(a)P_1(a)P_2(a)}[S(X,a) - 1]$$

Collecting terms in $f_1(0)$ and solving for $f_1(0)$ yields

$$f_1(0) = \frac{K(X) - \dfrac{c_2X^2}{P_2(a)} - \dfrac{[2c_2 + c_1P_2(a)]X}{P_1(a)P_2(a)}}{S(X,a) - 1}$$

$$+ \frac{2c_2 + c_1P_2(a) + c_0P_1(a)P_2(a)}{P_0(a)P_1(a)P_2(a)}$$

The analysis of the quadratic form of $\hat{g}(x)$ is now restricted to the case where the fixed cost of replacement $K(X)$ is independent of the replacement age, that is, $K(X) = K =$ constant. Then

$$f_1(0) = \frac{K - \dfrac{c_2X^2}{P_2(a)} - \dfrac{[2c_2 + c_1P_2(a)]X}{P_1(a)P_2(a)}}{S(X,a) - 1}$$

$$+ \frac{2c_2 + c_1P_2(a) + c_0P_1(a)P_2(a)}{P_0(a)P_1(a)P_2(a)} \tag{3.39}$$

For notational convenience let

$$\beta = \frac{c_2}{P_2(a)} \tag{3.40}$$

$$\gamma = \frac{2c_2 + c_1P_2(a)}{P_1(a)P_2(a)} \tag{3.41}$$

Using (3.40) and (3.41), equation (3.39) can be written as

$$f_1(0) = \frac{K - \beta X^2 - \gamma X}{S(X,a) - 1} + \frac{2c_2 + c_1P_2(a) + c_0P_1(a)P_2(a)}{P_0(a)P_1(a)P_2(a)} \tag{3.42}$$

The following theorem presents the results obtained for the quadratic form of $\hat{g}(x)$

Theorem 3.3

Sufficient conditions for a unique finite optimal equipment replace-

... age λ to exist for the linear hysteresis problem where $a(x) = \alpha x$, $(y) = \lambda$ = constant, $K(X) = K$ = constant, and $g(x) = c_2 x^2 + c_1 x + c_0$ ... re that $c_i > 0$ for $i=1,2$ and

$$(2\beta X + \gamma)[S(X,a) - 1] + (K - \beta X^2 - \gamma X)S_X(X,a) = 0 \qquad (3.43)$$

where λ is the unique solution to (3.43).

Proof

If the initial equipment age x is such that $x < X$, then it is clear from substituting (3.42) into (3.38) that if an optimal replacement age X exists, it minimizes both $f_1(0)$ and $f_1(x)$ for all x. Differentiating the expression for $f_1(0)$ in (3.42) with respect to X and equating the result to zero, a sufficient condition for the existence of an optimal equipment replacement age X, say X^*, is obtained and is given by the equation in X^*

$$(2\beta X^* + \gamma)[S(X^*,a) - 1] + (K - \beta X^{*2} - \gamma X^*)S_X(X^*,a) = 0 \qquad (3.44)$$

Note that (3.44) is identical to (3.43). Dividing (3.44) by $\gamma S_X(X^*,a)$ and rearranging yields

$$\frac{\beta}{\gamma} X^{*2} + X^* - \left[\frac{2\beta}{\gamma} X^* + 1\right]\left[\frac{S(X^*,a) - 1}{S_X(X^*,a)}\right] = \frac{K}{\gamma}$$

Letting $\delta = \beta/\gamma$ in the last expression, the following is obtained

$$\delta X^{*2} + X - (2\delta X^* + 1)\left[\frac{S(X^*,a) - 1}{S_X(X^*,a)}\right] = \frac{K}{\gamma} \qquad (3.45)$$

Taking the second derivative of $f_1(0)$ with respect to X yields

$$\frac{d^2 f_1(0)}{dX^2} = \frac{[-2\delta[S(X,a)-1] - (2\delta X+\gamma)S_X(X,a)][S(X,a)-1]^2}{[S(X,a)-1]^4}$$

$$+ \frac{\{-(-2\delta X-\gamma)S_X(X,a) - (K-\delta X^2-\gamma X)S_{XX}(X,a)\}[S(X,a)-1]^2}{[S(X,a)-1]^4}$$

$$\frac{2S_X(X,a)[S(X,a)-1]\{-(2\delta X+\gamma)[S(X,a)-1]\}}{[S(X,a)-1]^4}$$

$$-\frac{2S_X(X,a)[S(X,a)-1]\{-(K-\delta X^2-\gamma X)S_X(X,a)\}}{[S(X,a)-1]^4} \qquad (3.46)$$

Substituting (3.44) in (3.4d) and simplifying

$$\left.\frac{d^2 f_1(0)}{dX^2}\right|_{X=X^*} = \frac{-2\delta[S(X^*,a)-1] - (K-\delta X^{*2}-\gamma X^*)S_{XX}(X^*,a)}{[S(X^*,a)-1]^2} \qquad (3.47)$$

Returning to equation (3.45), let $M(X)$ equal the left-hand side of the equation, that is

$$M(X) = \delta X^2 + X - (2\delta X+1)\left[\frac{S(X,a)-1}{S_X(X,a)}\right]$$

Differentiating $M(X)$ with respect to X yields

$$\frac{dM(X)}{dX} = 2\delta X + 1 - 2\delta\left[\frac{S(X,a)-1}{S_X(X,a)}\right]$$

$$-(2\delta X+1)\left\{\frac{S_X(X,a)S_X(X,a) - S_{XX}(X,a)[S(X,a)-1]}{[S_X(X,a)]^2}\right\}$$

Simplifying

$$\frac{dM(X)}{dX} = -2\delta\left[\frac{S(X,a)-1}{S_X(X,a)}\right] + (2\delta X + 1)\left\{\frac{S_{XX}(X,a)[S(X,a)-1]}{[S_X(X,a)]^2}\right\}$$

$$-\left[(2\delta X + 1)S_{XX}(X,a) - 2\delta S_X(X,a)\right]\left[\frac{S(X,a)-1}{S_X(X,a)}\right] \tag{3.48}$$

Now using (3.14) and result (B.3) of Appendix B, $S_X(X,a)$ and $S_{XX}(X,a)$ can be written as

$$S_X(X,a) = P_0(a) + \sum_{j=1}^{\infty} \frac{X^j}{j!} \prod_{k=0}^{j} P_k(a) \tag{3.49}$$

$$S_{XX}(X,a) = P_0(a)P_1(a) + \sum_{j=1}^{\infty} \frac{X^j}{j!} \prod_{k=0}^{j+1} P_k(a) \tag{3.50}$$

Then

$$(2\delta X + 1)S_{XX}(X,a) - 2\delta S_X(X,a)$$

$$= 2\delta[XS_{XX}(X,a) - S_X(X,a)] + S_{XX}(X,a)$$

$$= 2\delta \sum_{j=2}^{\infty} \frac{(j-1)X^j}{j!} \prod_{k=0}^{j} P_k(a) - 2\delta P_0$$

$$+ P_0(a)P_1(a) + \sum_{j=1}^{\infty} \frac{X^j}{j!} \prod_{k=0}^{j+1} P_k(a) \tag{3.51}$$

Since $\delta = \beta/\gamma$, then using (3.40) and (3.41) δ can be expressed as

$$\delta = \frac{c_2 P_1(a)}{2c_2 + c_1 P_2(a)}$$

Substituting the expression obtained for δ into (3.51) yields

$$(2\delta X + 1)S_{XX}(X,a) - 2\delta S_X(X,a)$$

$$= \frac{2c_2 P_1(a)}{2c_2 + c_1 P_2(a)} \sum_{j=2}^{\infty} \frac{(j-1)X^j}{j!} \prod_{k=0}^{j} P_k(a)$$

$$+ \sum_{j=1}^{\infty} \frac{X^j}{j!} \sum_{k=0}^{j+1} P_k(s)$$

$$+ P_0(s)P_1(s) \left[1 - \frac{2c_2}{2c_2 + c_1 P_2(s)}\right]$$

If $c_i > 0$ for $i=1,2$, then

$$0 < \frac{2c_2}{2c_2 + c_1 P_2(s)} < 1$$

which implies that

$$(2\delta X + 1)S_{XX}(X,s) - 2\delta S_X(X,s) > 0 \quad \text{for all } X \geq 0 \qquad (3.52)$$

Using (3.52) in (3.48), it follows that

$$\frac{dM(X)}{dX} > 0 \quad \text{for all } X > 0$$

Furthermore, $M(0) = 0$. Therefore the function $M(X)$ is positive for $X > 0$ and is strictly increasing to infinity. If $c_i > 0$ for $i=1,2$, then

$$\frac{kP_1(s)P_2(s)}{2c_2 + c_1 P_2(s)} > 0$$

It therefore follows from (3.45) that the optimum equipment replacement age X, if it exists, is unique.

The existence of an optimal equipment replacement age X, which minimizes $f_1(0)$, is established by proving that $\left.\frac{d^2 f_1(0)}{dX^2}\right|_{X=X^*} > 0$. Equation (3.44) can be rearranged to yield

$$X \quad \delta X^{*2} - \gamma X^* = \quad (2\delta X^* + \gamma)\left[\frac{S(X^*,s) - 1}{S_X(X^*,s)}\right] \qquad (3.53)$$

Substituting (3.53) in (3.77) yields

$$\left.\frac{d^2 f_1(0)}{dX^2}\right|_{X=X^*} = \frac{-2\delta S_X(X^*,a) + (2\delta X^* + \gamma) S_{XX}(X^*,a)}{S_X(X^*,a)[S(X^*,a) - 1]}$$

Since $\delta = B/\gamma$, then

$$\left.\frac{d^2 f_1(0)}{dX^2}\right|_{X=X^*} = \frac{(2\delta X^* + 1) S_{XX}(X^*,a) - 2\delta S_X(X^*,a)}{\gamma S_X(X^*,a)[S(X^*,a) - 1]}$$

Using (3.52) in the last expression, it follows that

$$\left.\frac{d^2 f_1(0)}{dX^2}\right|_{X=X^*} > 0$$

This proves existence and concludes the proof of Theorem 3.3. The condition $c_i > 0$ for i=1,2 is one of a pair of sufficient conditions for a unique finite optimal equipment replacement age X to exist.

The following is a numerical example of the linear hysteresis problem with $g(x) = c_2 x^2 + c_1 x + c_0$.

<u>Example 2</u>

Consider the problem posed in Example 1, with one exception. The combined repair and operating cost in \$ per year of service for the equipment when service age is u is $1.73\ u^2 + 7.62\ u + 6.5$.

From the problem statement, it is known that K = \$350, λ = .1/year, ℓ = .1\$/(\$year), B = .2/year, a = 1/2, $\overline{h}_T(1) = 1$, $\lambda\overline{h}_T(1)$ = .1/year, $P_1(1/2)$ = .15, $c_2 = 1.73$, $c_1 = 7.62$, and $c_0 = 6.5$. Then solving numerically equation (3.43), it is determined that the

[illegible] an equipment should be replaced is 6.39 years.

The quadratic cost function $g(x) = 1.75\,x^2 + 7.5\,x$ is examined in Tables 3.6 through 3.10 for specific values of a. These values are 0, 0.25, 0.50, 0.75 and 1. Each table presents the optimum economic life in years and the corresponding value for $f_1(0)$ for various combinations of K, R, and $\bar{A}\bar{K}_T(t)$.

The behavior of X and $f_1(0)$ observed from Tables 3.6 through 3.10 for the quadratic cost function is the same as for the linear cost function previously discussed, and therefore will not be repeated here.

It should be noted that the sufficient condition on the coefficients of the polynomial g(x) for a unique finite optimal equipment replacement age X to exist is not a necessary condition. There may exist a polynomial g(x) with some negative coefficients which is convex over an interval which contains a unique optimal age X.

Table 3.6 Table of Optimum X and Corresponding $f_1(0)$ for

$\hat{g}(x) = 1.75x^2 + 7.5x$ When $a = 0.0$

ϕ	$\lambda\bar{h}_2(1)$	K = 50: X	K = 50: $f_1(0)$	K = 150: X	K = 150: $f_1(0)$	K = 250: X	K = 250: $f_1(0)$	K = 350: X	K = 350: $f_1(0)$
0.20	0.10	2.91	267.6	4.71	441.3	5.88	547.3	6.82	[illegible]
0.40	0.10	3.17	71.3	5.36	101.2	6.87	113.9	8.12	121.0
	0.20	3.17	107.0	5.36	151.8	6.87	170.9	8.12	181.4
	0.30	3.17	214.0	5.36	303.5	6.87	341.8	8.12	362.9
0.60	0.10	3.46	33.8	6.10	41.5	7.99	43.3	9.57	43.9
	0.20	3.46	42.3	6.10	51.8	7.99	54.1	9.57	54.9
	0.30	3.46	56.3	6.10	69.1	7.99	72.2	9.57	73.2
	0.40	3.46	84.5	6.10	103.7	7.99	108.3	9.57	109.9
	0.50	3.46	169.0	6.10	207.3	7.99	216.5	9.57	219.7
0.80	0.10	3.78	19.0	6.89	20.9	9.14	21.2	11.01	21.2
	0.20	3.78	22.2	6.89	24.4	9.14	24.7	11.01	24.7
	0.30	3.78	26.6	6.89	29.3	9.14	29.6	11.01	29.7
	0.40	3.78	33.3	6.89	36.7	9.14	37.0	11.01	37.1
	0.50	3.78	44.4	6.89	48.9	9.14	49.4	11.01	49.4
	0.60	3.78	66.6	6.89	73.3	9.14	74.0	11.01	74.2
	0.70	3.78	133.2	6.89	146.6	9.14	148.1	11.01	148.3

Table 3.7 Table of Optimum X and Corresponding $f_1(0)$ for $g(x) = 1.75x^2 + 7.5x$ When $a = 0.25$

		K							
		50		150		250		350	
n	$\lambda \bar{k}_T(i)$	X	$f_1(0)$	X	$f_1(0)$	X	$f_1(0)$	X	$f_1(0)$
0.20	0.10	2.86	276.4	4.58	464.1	5.71	581.9	6.60	670.[illegible]
0.40	0.10	3.12	73.6	5.24	106.1	6.72	120.4	7.93	128.5
	0.20	3.06	114.1	5.12	167.1	6.55	191.2	7.74	205.1
	0.30	3.00	235.7	5.00	351.2	6.39	405.7	7.53	437.9
0.60	0.10	3.41	34.9	6.00	43.2	7.86	45.3	9.42	46.0
	0.20	3.35	44.9	5.89	56.3	7.73	59.3	9.27	60.3
	0.30	3.30	61.8	5.78	78.5	7.59	82.9	9.11	84.5
	0.40	3.24	95.7	5.67	123.1	7.45	130.6	8.95	133.5
	0.50	3.19	197.7	5.56	257.8	7.30	275.1	8.77	281.7
0.80	0.10	3.73	19.6	6.80	21.7	9.04	21.9	10.90	21.9
	0.20	3.68	23.5	6.71	26.2	8.93	26.5	10.78	26.5
	0.30	3.63	29.0	6.62	32.5	8.82	32.9	10.65	33.0
	0.40	3.58	37.4	6.53	42.2	8.71	42.8	10.53	42.9
	0.50	3.52	51.3	6.43	58.4	8.59	59.3	10.39	59.5
	0.60	3.47	79.4	6.33	91.0	8.47	92.6	10.26	92.9
	0.70	3.42	163.8	6.23	189.5	8.34	193.0	10.11	193.8

Table 3.8 Table of Optimum X and Corresponding $f_1(0)$ for $g(x) = 1.75x^2 + 7.5x$ When $a = 0.50$

		K							
		50		150		250		350	
s	$\lambda\bar{h}_T(1)$	X	$f_1(0)$	X	$f_1(0)$	X	$f_1(0)$	X	$f_1(0)$
0.28	0.10	2.81	284.7	4.48	487.2	5.34	618.5	6.38	719.6
0.40	0.10	3.07	76.0	5.12	111.6	6.54	128.1	7.70	137.8
	0.20	2.96	121.4	4.88	185.2	6.19	217.3	7.26	237.3
	0.30	2.86	258.9	4.63	410.9	5.83	493.9	6.81	548.9
0.60	0.10	3.35	36.0	5.87	45.3	7.69	47.7	9.21	48.7
	0.20	3.25	47.9	5.64	62.1	7.37	66.2	8.82	67.9
	0.30	3.14	68.1	5.40	91.3	7.03	98.7	8.42	101.8
	0.40	3.03	109.0	5.15	151.5	6.68	166.7	7.98	173.6
	0.50	2.93	232.9	4.89	336.7	6.31	378.3	7.52	398.8
0.80	0.10	3.67	20.2	6.68	22.5	8.88	22.8	10.71	22.9
	0.20	3.57	25.0	6.47	28.4	8.61	28.8	10.39	28.9
	0.30	3.46	31.9	6.25	37.0	8.32	37.7	10.06	37.9
	0.40	3.35	42.5	6.02	50.4	8.02	51.7	9.70	52.0
	0.50	3.24	60.4	5.78	73.6	7.70	76.0	9.33	76.6
	0.60	3.13	96.8	5.53	121.6	7.36	126.8	8.93	128.2
	0.70	3.02	207.2	5.27	269.5	7.00	284.3	8.49	288.7

Table 3.9 Table of Optimum X and Corresponding $f_1(0)$ for

$g(x) = 1.75x^2 + 7.5x$ When $a = 0.75$

		K							
		50		150		250		[illegible]	
ℓ	$\lambda\bar{h}_T(\ell)$	X	$f_1(0)$	X	$f_1(0)$	X	$f_1(0)$	X	[illegible]
0.20	0.10	2.80	291.9	4.61	508.4	5.45	653.2	6.25	[illegible]
0.40	0.10	3.04	78.1	5.03	117.3	6.38	136.6	7.49	148.1
	0.20	2.91	128.5	4.71	204.3	5.90	246.9	6.87	273.9
	0.30	2.78	280.7	4.40	473.9	5.44	595.4	6.26	686.2
0.60	0.10	3.32	37.1	5.75	47.7	7.50	50.7	8.96	51.9
	0.20	3.17	50.9	5.41	69.0	7.00	75.3	8.32	78.2
	0.30	3.04	74.5	5.06	107.0	6.48	120.7	7.67	127.8
	0.40	2.90	122.7	4.73	187.5	5.97	219.5	7.00	238.9
	0.50	2.77	269.1	4.40	438.0	5.48	534.9	6.34	601.3
0.80	0.10	3.63	20.8	6.55	23.6	8.69	23.9	10.47	24.0
	0.20	3.48	26.6	6.21	31.2	8.21	32.0	9.89	32.2
	0.30	3.33	35.1	5.85	43.0	7.71	44.7	9.28	45.1
	0.40	3.18	48.3	5.50	62.2	7.19	65.9	8.64	67.1
	0.50	3.04	70.8	5.14	96.8	6.66	105.3	7.96	108.8
	0.60	2.90	116.9	4.79	170.3	6.12	192.2	7.25	203.0
	0.70	2.77	257.2	4.44	400.8	5.58	472.4	6.54	514.6

Table 3.10 Table of Optimum X and Corresponding $f_1(0)$

$g(x) = 1.75x^2 + 7.5x$ When $a = 1.0$

		K							
		50		150		250		350	
s	$\lambda\bar{K}_T(\lambda)$	X	$f_1(0)$	X	$f_1(0)$	X	$f_1(0)$	X	$f_1(0)$
0.20	0.10	2.80	287.4	4.43	525.0	5.46	681.1	6.27	806.4
0.40	0.10	3.04	79.8	5.02	122.4	6.36	344.6	7.44	359.3
	0.20	2.91	135.8	4.71	220.6	5.88	275.6	6.82	312.7
	0.30	2.80	287.4	4.43	525.0	5.46	681.1	6.27	806.4
0.60	0.10	3.31	38.1	5.72	50.1	7.42	54.0	8.84	55.8
	0.20	3.17	53.5	5.36	75.8	6.87	85.4	8.12	80.7
	0.30	3.04	79.8	5.02	122.4	6.36	344.6	7.44	158.1
	0.40	2.91	135.8	4.71	220.6	5.88	273.6	6.82	312.7
	0.50	2.80	297.4	4.43	525.0	5.46	681.1	6.27	806.4
0.80	0.10	3.62	21.4	6.48	24.7	8.57	25.3	10.50	25.4
	0.20	3.46	28.2	6.10	34.6	7.88	36.1	9.57	36.6
	0.30	3.31	38.1	5.72	50.1	7.42	54.0	8.84	55.8
	0.40	3.17	53.5	5.36	75.8	6.87	85.4	8.12	90.7
	0.50	3.04	79.8	5.02	322.4	6.36	144.6	7.44	158.1
	0.60	2.91	133.8	4.71	220.6	5.88	275.6	6.82	312.7
	0.70	2.80	287.4	4.43	525.0	5.46	681.1	6.27	806.4

CHAPTER 4

AGE REPLACEMENT WITH ECONOMICALLY CONTROLLED PARTIAL REPAIR AND DOWNTIME

4.1 Introduction

In this chapter, the linear hysteresis problem examined in Chapter 3 is extended to incorporate the problem of economically controlling the amount of partial repair following a failure. This requires the addition of several new concepts. In addition to the repair cost $k(u)$ incurred at failure for an equipment aged u, $C(E,u)$ is spent at the commencement of the repair, where E is the level of expenditure. The expenditure repair function $C(E,u)$ is defined to be equal to Eu^m, where m is some non-negative integer. If E equals zero, then no additional funds are expended on the repair, and a minimal repair is performed. For E greater than zero, service age can be recovered at the termination of the repair. If $C(E,u)$ is spent on the repair of an equipment aged u, the fraction of service age remaining after the repair is $p(E)$, where $0 \leq p(E) \leq 1$. Then the service age at the end of the repair is $p(E)u$. The function $p(E)$ corresponds to the constant a of the linear hysteresis age function, $a(x)$ equals ax.

Some possible shapes for the curve (function) $p(E)$ are illustrated in Figures 4.1 through 4.6. In each of the figures, the curve $p(E)$ is strictly non-increasing in E. The linear curve $P(E)$ in Figure 4.1 is a special case of the curve in Figure 4.2 with $L_2 = 0$ and $L_1 = L_3$. Figure 4.2 is an example of an initial expenditure threshold where an amount L_2 must be expended before any decrease in $p(E)$ is noticed. A similar type of threshold is observed in Figure 4.3.

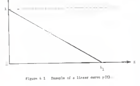

Figure 4 1 Example of a linear curve p(E)

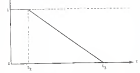

Figure 4 2 Example of a linear curve y(E)

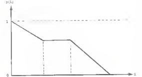

Figure 4.3 Example of a curve p(E)

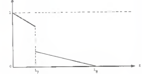

Figure 4.4 Example of a curve p(E)

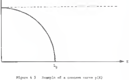

Figure 4.5 Example of a concave curve p(E)

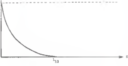

Figure 4.6 Example of a convex curve p(E)

increasing the level of expenditures beyond L_4 and up to L_5 does not result in a decrease in p(E). In Figure 4.4 a different threshold occurs. At L_7 there is a jump in the value of p(E) without any corresponding change in E. Figures 4.5 and 4.6 are examples of piecewise non-linear curves p(E) which are concave and convex, respectively.

The value of E acts as a decision variable since it controls both C(E,u) and p(E). Consequently, the age replacement policy of Chapters 2 and 3 is modified to incorporate E, and becomes the two-variable decision policy, the E,X-policy.

The new problem generated from extending the linear hysteresis problem so that partial repair is economically controlled is called the economically controlled linear hysteresis problem. This problem with non-zero downtime for repair is examined in this chapter. For the functional-differential equation (1.14) with failure rate function, (x), equal to , and with p(E) equal to a ($0 \leq a \leq 1$), a constant, a general solution is determined. The method of obtaining this solution is identical to the method discussed in the introduction to Chapter 3. For a constant operating cost and a constant expenditure repair function C(E,u), a pair of sufficient conditions is determined for which an E,X-policy is an optimal policy for the economically controlled linear hysteresis problem. For several curves of p(E) versus E, the E,X-policy is investigated for an equipment subject to a linear operating cost and a constant or linear expenditure repair function C(E,u).

4.2 General Solution to Functional-Differential Equation with $\lambda(x) = \lambda$ and $p(E) = a$

The functional-differential equation (1.14) of Chapter 1 is

multiplied, we get

$$f_1'(x) - [1 + \lambda(x)]f_1(x) + \lambda(x)\overline{h}_T(1)f_1(p(E)x) = -q(x) \qquad (4.1)$$

where

$$q(x) = c(x) + k(x)\lambda(x) + \frac{r(x)}{1}[1 - \overline{h}_T(1)]\lambda(x) + C(E,x)\lambda(x) \qquad (4.2)$$

Letting $p(E) = a$ (a constant to be determined later, where $0 \leq a \leq 1$) in (4.1) and $\lambda(x) = \lambda$ = constant in (4.1) and (4.2) yields

$$f_1'(x) - (1 + \lambda)f_1(x) + \lambda\overline{h}_T(1)f_1(ax) = -\hat{q}(x) \qquad (4.3)$$

$$\hat{q}(x) = c(x) + \lambda k(x) + \frac{\lambda r(x)}{1}[1 - \overline{h}_T(1)] + \lambda C(E,x) \qquad (4.4)$$

The procedure to obtain the general solution to (4.3) consists of determining a complementary solution, a particular solution, and the constant term of these solutions by evaluating the general solution at x equals zero. It can be verified (Appendix A) that the general solution to (4.3) is of the form

$$f_1(x) = y_1(x) + y_2(x) \qquad (4.5)$$

where $y_1(x)$ is a solution to the homogeneous equation

$$y_1'(x) - (1 + \lambda)y_1(x) + \lambda\overline{h}_T(1)y_1(ax) = 0 \qquad (4.6)$$

and $y_2(x)$ is a solution to the non-homogeneous equation

$$y_2'(x) - (1 + \lambda)y_2(x) + \lambda\overline{h}_T(1)y_2(ax) = -\hat{q}(x) \qquad (4.7)$$

A complementary solution to the homogeneous equation (4.6) has been previously obtained in Chapter 3 (see (3.6) and (3.8) through (3.16)). It is

$$y_1(x) = a_0 E(x,a) \tag{4.8}$$

where a_0 is an arbitrary constant to be determined.

To determine a particular solution $y_2(x)$, it is necessary to solve the linear differential-difference equation (4.7). It is observed that the term $y_2(ax)$ in equation (4.7) constitutes a backward differencing, since $0 \leq a \leq 1$. The effect of this term in equation (4.7) requires an assumption about the nature of $\hat{q}(x)$ in order to obtain $y_2(ax)$. Observe from (3.4) and (4.4) that

$$\hat{q}(x) = \hat{g}(x) + \lambda C(E,x) \tag{4.9}$$

As in Chapter 3, $\hat{g}(x)$ is assumed to be a member of the class of nth degree polynomials whose coefficients are all required to be positive. Then, $\hat{g}(x)$ is a polynomial of the form

$$\hat{g}(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where

$$c_i > 0 , \quad i=0,1,\ldots,n$$

From (1.11), $C(E,x)$ is given by

$$C(E,x) = Ex^m , \quad m=0,1,2,3\ldots$$

Assuming that $m \leq n$, the expressions for $\hat{g}(x)$ and $C(E,x)$ are substituted in (4.9) to obtain

$$\begin{aligned} \hat{q}(x) = a_n x^n + \cdots + a_{m+1}x^{m+1} + (a_m + \lambda E)x^m \\ + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0, \quad m=0,1,\ldots,n \end{aligned} \tag{4.10}$$

Given the assumption on the polynomial $\hat{g}(x)$ and since $E \geq 0$, $\hat{q}(x)$ is

also a member of the class of mth degree polynomials whose coefficients are all required to be positive. Therefore, $\hat{q}(x)$ is a convex function. Let

$$\hat{a}_i = \begin{cases} a_i\ , & i=0,1,\ldots,m-1,m+1,\ldots,n \\ a_i + \lambda E & i=m \end{cases} \tag{4.11}$$

Using (4.11) in (4.10), $\hat{q}(x)$ can be written as

$$\hat{q}(x) = \hat{a}_n x^n + \hat{a}_{n-1} x^{n-1} + \cdots + \hat{a}_1 x + \hat{a}_0$$

where

$$\hat{a}_i > 0\ , \quad i=0,1,\ldots,n \tag{4.12}$$

Then the particular solution $y_2(x)$ to (4.7) with $\hat{q}(x)$ given by (4.12) is

$$y_2(x) = d_n x^n + d_{n-1} x^{n-1} + \cdots + d_1 x + d_0 \tag{4.13}$$

Applying results obtained in Chapter 3 for the particular solution in order to evaluate the d_i $(i=0,1,\ldots,n)$ in (4.13), it is determined that the particular solution is

$$y_2(x) = \frac{a_n x^n}{F_n(a)} + \frac{[a_{n-1}F_{n-1}(a) + na_n]x^{n-1}}{F_{n-1}(a)F_n(a)} + \sum_{j=0}^{n-2} x^j \left\{ \frac{a_j}{F_j(a)} + \frac{1}{\prod\limits_{k=j}^{n} F_k(a)} \left[\sum_{i=j+1}^{n-1} (j+1)\cdots(i)a_i \prod_{k=i+1}^{n} F_k(a) + (j+1)\cdots(n)\hat{a}_n \right] \right\} \tag{4.14}$$

which is a solution to (4.7) with the mth degree polynomial $\hat{q}(x)$ given by (4.12).

Thus, the general solution to (4.3) with $\hat{q}(x)$ given by (4.12) is

$$
\begin{aligned}
f_1(x) &= y_1(x) + y_2(x) \\
&= a_0 S(x,a) + \frac{\hat{a}_n x^n}{P_n(a)} + \frac{[\hat{a}_{n-1}P_{n-1}(a) + n\hat{a}_n]x^{n-1}}{P_{n-1}(a)P_n(a)} \\
&\quad + \sum_{j=0}^{n-2} x^j \left\{ \frac{\hat{a}_j}{P_j(a)} + \frac{1}{\prod_{k=j}^{n} P_k(a)} \left[\sum_{i=j+1}^{n-1} (j+1)\cdots(i)\hat{a}_i \prod_{k=i+1}^{n} P_k(a) \right.\right. \\
&\quad \left.\left. + (j+1)\cdots(n)\hat{a}_n \right] \right\}
\end{aligned}
\tag{4.15}
$$

To evaluate the arbitrary constant a_0, recall that for $x = 0$ in (3.14)

$$S(0,a) = 1$$

Evaluating (4.15) at $x = 0$ to solve for a_0 and substituting the value obtained into (4.15) yields

$$
\begin{aligned}
f_1(x) &= f_1(0)S(x,a) + \frac{\hat{a}_n x^n}{P_n(a)} + \frac{[\bar{a}_{n-1}P_{n-1}(a) + n a_n]x^{n-1}}{P_{n-1}(a)P_n(a)} \\
&\quad + \sum_{j=0}^{n-2} x^j \left\{ \frac{\hat{a}_j}{P_j(a)} + \frac{1}{\prod_{k=j}^{n} P_k(a)} \left[\sum_{i=j+1}^{n-1} (j+1)\cdots(i)a_i \prod_{k=i+1}^{n} P_k(a) \right.\right. \\
&\quad \left.\left. + (j+1)\cdots(n)a_n \right] \right\} - \frac{1}{\prod_{k=0}^{n} P_k(a)} \left[\sum_{i=0}^{n-1} (i!)a_i \prod_{k=i+1}^{n} P_k(a) \right. \\
&\quad \left. + (n!)a_n \right][S(x,a) - 1]
\end{aligned}
\tag{4.16}
$$

Thus, if $\bar{q}(x)$ is an nth degree polynomial, then equation (4.16) is the general solution to the functional-differential equation (4.3). The following section applies (4.16) in the discussion of three particular polynomials $g(x)$.

4.3 Special Polynomial Forms of $q(x)$

In this section, several polynomial forms of $\hat{q}(x)$ are investigated in conjunction with curves $p(R)$ in order to determine optimal values for the parameters of the E,X-policy. It is assumed that all the coefficients of the polynomial $\hat{q}(x)$ are positive.

a. $q(x) = c_0 + \lambda E$

Recall that in the linear hysteresis problem with $\hat{g}(x) = c_0$, there was never an incentive to replace, regardless of the value of a, and further it was established that the optimal replacement age was infinite.

Letting $q(x) = c_0 + \lambda E$ in (4.3) yields the functional-differential equation

$$f_1'(x) - (1 + \lambda)f_1(x) + \lambda\bar{h}_2(1)f_2(ax) = - c_0 - \lambda E \tag{4.17}$$

Applying (4.16) which is the general solution to (4.3), the general solution to (4.17) can be written as

$$f_1(x) = f_1(0)S(x,a) - \frac{c_0 + \lambda E}{P_0(a)}\left[S(x,a) - 1\right] \tag{4.18}$$

If an optimal E,X-policy is followed, continuity of the function $f_1(x)$ at $x = X$ is required. It then follows that

$$f_1(X) = K(X) + f_1(0) \tag{4.19}$$

Applying (4.19) in (4.18), it is required that at $x = X$

$$K(X) + f_1(0) = f_1(0)S(X,a) - \frac{c_0 + \lambda E}{P_0(a)}\left[S(X,a) - 1\right]$$

Solving for $f_1(0)$ yields

$$f_1(0) = \frac{K(X)}{S(X,s) - 1} + \frac{c_0 + \lambda E}{P_0(s)} \qquad (4.20)$$

Consider an arbitrary function $p(E)$. By assumption, $K(X)$ is a continuous and bounded function of X. For each value of s (and corresponding value of E since $p(E) = s$), $S(X,s)$ is strictly increasing in X to infinity. Thus, $f_1(0)$ is minimized with respect to X and E as $X \to \infty$ and for $E = 0$. The following theorem summarizes the result.

Theorem 4.1

Sufficient conditions for an E,X-policy to be an optimal policy for the economically controlled linear hysteresis problem where $p(E)$ is an arbitrary function, $\lambda(s) = \lambda$ = constant, and $\bar{q}(x) = c_0 + \lambda E$ are that $E = 0$ and X be infinite.

b. $q(x) = c_1 s + c_0 + \lambda E$

Let

$$q(x) = c_1 x + c_0 + \lambda E \qquad (4.21)$$

Substituting (4.21) in (4.3) yields the functional-differential equation

$$f_1'(x) - (1 + \lambda)f_1(s) + \lambda \bar{h}_p(1) f_1(ax) = - c_1 x - c_0 - \lambda E \qquad (4.22)$$

Applying (4.16), which is the general solution to (4.5), the general solution to (4.22) can be written as

$$f_1(s) = f_1(0)S(s,s) + \frac{c_1 x}{P_1(s)} - \frac{c_1 + (c_0 + \lambda E)P_1(s)}{P_0(s)P_1(s)} [S(s,s) - 1] \qquad (4.23)$$

Using (4.19) in (4.23), it follows that at $s = X$

$$K(X) + f_1(0) = f_1(0)S(X,s) + \frac{c_1 X}{P_1(s)} - \frac{c_1 + (c_0 + \lambda E)P_1(s)}{P_0(s)P_1(s)} [S(X,s) - 1]$$

Solving for $f_1(0)$ yields

$$f_1(0) = \frac{K(X) - \frac{c_1 X}{P_1(a)}}{S(X,a) - 1} + \frac{c_1 + (c_0 + \lambda E)P_1(a)}{P_0(a)P_1(a)} \tag{4 24}$$

For the remainder of the discussion of the polynomial (4 21), the analysis is restricted to the situation where the fixed cost of replacement K(X) is independent of the replacement age X. Since K(X) - X = constant, then

$$f_1(0) = \frac{K - \frac{c_1 X}{P_1(a)}}{S(X,a) - 1} + \frac{c_1 + (c_0 + \lambda E)P_1(a)}{P_0(a)P_1(a)} \tag{4 25}$$

For the polynomial (4 21), neither necessary nor sufficient conditions for an optimal E,X-policy to exist can be obtained This is due to the various shapes the curve p(E) can assume (see Figures 4 1 through 4 6) and the structure of the expression obtained for $f_1(0)$ in (4 25) Although the derivative of S(X,a) with respect to a exists, it is not possible to obtain a closed form expression for the derivative of the curve p(E), especially at the end points of the interval over which the curve p(E) is defined Through another method of optimization, it is determined that for some curves p(E) and expenditure repair functions C(E,u), the value of E of the pair E,X, which minimizes $f_1(0)$, corresponds to an end point of the curve p(E)

Since analytic results for the polynomial (4 21) cannot be determined, quantitative results are obtained. In Chapter 3 for an equipment subject to a linear operating cost $\hat{g}(a)$ (i.e., $\hat{g}(x) = c_1 x + c_0$), a pair of sufficient conditions are determined.

Similar conditions can be determined for the polynomial (4.23). For this polynomial, the conditions are $c_1 > 0$ and

$$\frac{c_1}{P_1(a)}\left[S(X,a) - 1\right] + \left[K - \frac{c_1 X}{P_1(a)}\right] S_X(X,a) = 0 \qquad (4.26)$$

which are identical to those determined for the linear operating cost $g(x)$. By specifying a value of a which corresponds to a value of E since $p(E) = a$, it is possible to obtain an optimal X (assuming that $c_1 > 0$) by utilizing the equation (4.26) in a search procedure. The search procedure is performed until all possible values of a have a corresponding optimal X. The function $f_1(0)$ is then evaluated for each of the pairs a, X, and the minimum is obtained.

For the polynomial (4.21), several basic curves $p(E)$ are examined, some of which are displayed in Figures 4.1 through 4.6. In order to generate a family of curves $p(E)$ for each basic curve, each basic curve is normalized with respect to the E axis and then multiplied by different scaling factors. The scaling factor corresponds to the value of E for which $p(E) = 0$. In addition, critical variables of the economically neutralized linear hysteresis problem for the polynomial (4.21) are varied to determine their effect on the optimal values of the parameters determined for the E,X-policy. These variables are K, c_1, δ (where $\delta = i + \lambda$), λ, and $\overline{H}_2(1)$. The following discussion presents the analysis of the quantitative results obtained.

For the concave curve $p(E)$ displayed in Figure 4.5, the results indicate that either minimal repair or major repair is performed following an equipment failure. In addition, as the scaling factor is

increased, there is a shift in repair effort from major repair to minimal repair. From the computational results obtained, the conclusion can be inferred that for all concave curves p(E) (including linear curves), regardless of the values of the critical parameters, only major or minimal repair is performed. For linear curves p(E) as in Figures 4.1 and 4.2, convex curves p(E) as in Figure 4.6, and more complex versions of the curves p(E) displayed in Figures 4.3 and 4.4, the amount of repair following an equipment failure can be any amount of partial repair from minimal to major repair, inclusive.

The following comments concerning the sensitivity of the parameters for the E,X-policy, as the critical variables are varied one at a time, apply to all the curves p(E) investigated. If either K increases, or λ increases, or c_1 decreases, the amount of partial repair and corresponding reduction of service age following a failure increases, and the replacement age X decreases. In addition, the value of $f_1(0)$, the minimum total expected discounted cost over an infinite time horizon starting with an equipment age zero, increases. If θ increases, the amount of partial repair decreases, and the replacement age X increases. Also, the value of $f_1(0)$ decreases. Decreasing $\overline{\Xi}_2(i)$, the Laplace-Stieltjes transform with parameter i of the repair time distribution function $H_2(\cdot)$, results in a decrease in the reduction of service age at the conclusion of a repair. Increasing the scaling factor results in a decrease in both the amount of partial repair and the replacement age X. However, the value of $f_1(0)$ increases.

The comments concerning the sensitivity of the parameters for the E,X-policy are illustrated in Table 4.1 for a convex curve p(E), as in Figure 4.6. For a concave curve p(E) as in Figure 4.5, Table 4.2

Table 4 1 Table of Optimum Values for E,X-po[illegible]
a Convex Curve p(E) as in Figure 4 6

K	q	λ	$\lambda\bar{h}_I(i)$	Scaling Factor	$q(x) = 10x + \lambda E$: E	X	a	$f_\lambda(0)$
100	0 6	0 1	0 1	30	6.75	7.13	0.55	37 44
100	0.6	0.3	0 3	30	11.40	8.50	0.40	78.89
200	0 2	0.1	0.1	35	35 00	8.00	0.00	433.81
200	0.4	0.1	0.1	35	28.52	10.26	0.10	93.53
300	0 2	0 1	0.1	40	48 00	10 37	0.00	477 18
400	0 2	0 1	0.1	40	40 00	12 80	0.00	499.75
400	0 8	0 3	0 3	20	14 60	23 87	0.15	74 88
400	0.6	0 3	0.3	25	14.37	22.87	0.25	77.97
300	0.8	0.3	0.3	30	15 15	16 97	0.30	80 51
300	0.6	0.45	0.3	30	9.75	15 82	0.45	86.30

Table 4.2 Table of Optimum Values for S,X-policy for a Concave Curve p(S) as in Figure 4.5

					$\hat{q}(x) = 10x + \lambda S$			
K	α	λ	$\lambda\bar{N}_T(1)$	Scaling Factor	S	X	a	$f_1(0)$
100	0.4	0.1	0.1	40	40.00	6.30	0.00	89.96
100	0.4	0.1	0.1	45	0.00	5.74	1.00	91.24
100	0.4	0.3	0.3	50	50.00	8.30	0.00	379.80
100	0.4	0.3	0.3	55	0.00	4.83	1.00	383.10
100	0.8	0.3	0.3	35	35.00	7.65	0.00	89.99
100	0.8	0.3	0.3	40	0.00	5.74	1.00	91.24
200	0.8	0.3	0.3	45	45.00	13.87	0.00	100.54
200	0.6	0.3	0.3	50	0.00	9.12	1.00	103.90
200	0.6	0.45	0.3	30	30.00	13.87	0.00	100.54
200	0.6	0.45	0.3	35	0.00	9.12	1.00	103.90

[illegible] the change from major repair to minimal repair as the variables are varied one at a time It should be noted that only the critical scaling factors at which the switch in the amount of partial repair occurs are indicated The majority of the results obtained for linear curves as in Figure 4.1 indicate that either minimal or major repair is performed following a failure. Two exceptions are illustrated in Table 4.3

c) $\underline{\bar{q}(x) = c_1 x + c_0 + \lambda E x}$

Let

$$\bar{q}(x) = c_1 x + c_0 + \lambda E x \tag{4.27}$$

Substituting (4.27) in (4.3) yields the functional-differential equation

$$f_1'(x) - (1 + \lambda) f_1(x) + \lambda \bar{F}_Y(1) f_1(ax) = - c_1 x - c_0 - \lambda E x \tag{4.28}$$

Applying (4.16), which is the general solution to (4.3), the general solution to (4.28) can be written as

$$f_1(x) = f_1(0) S(x,a) + \frac{c_1 x + \lambda E x}{F_1(a)} - \frac{c_1 + \lambda E + c_0 F_1(a)}{F_0(a) F_1(a)} [S(x,a) - 1] \tag{4.29}$$

Applying (4.19) in (4.29), it follows that at $x = X$

$$K(X) + f_1(0) = f_1(0) S(X,a) + \frac{c_1 X + \lambda E X}{F_1(a)} - \frac{c_1 + \lambda E + c_0 F_1(a)}{F_0(a) F_1(a)} [S(X,a) - 1]$$

Solving for $f_1(0)$ yields

$$f_1(0) = \frac{K(X) - \dfrac{c_1 X + \lambda E X}{F_1(a)}}{S(X,a) - 1} + \frac{c_1 + \lambda E + c_0 F_1(a)}{F_0(a) F_1(a)} \tag{4.30}$$

Letting $K(X) = K$ = constant in (4.30), the following is obtained

Table 4.2 Table of Optimum Values for E,X-policy [illegible] Linear Curve p(E) as in Figure 4.1

K	δ	λ	$\lambda \bar{h}_z(1)$	Scaling Factor	$q(x) = 10x + \varepsilon E$: E	X	ε	$f_1(0)$
100	0.6	0.3	0.3	25	25.00	7.65	0.00	79.99
100	0.6	0.3	0.3	30	25.50	7.20	0.15	84.69
100	0.6	0.3	0.3	35	22.75	6.63	0.35	88.45
100	0.6	0.3	0.3	40	0.00	5.74	1.00	91.24
200	0.6	0.3	0.3	25	25.00	13.67	0.00	80.54
200	0.6	0.3	0.3	30	27.00	13.07	0.10	85.46
200	0.6	0.3	0.3	35	28.00	12.47	0.20	89.69
200	0.6	0.3	0.3	40	26.00	11.57	0.35	93.25
200	0.6	0.3	0.3	45	24.75	10.99	0.45	96.28
200	0.6	0.3	0.3	50	22.50	10.42	0.55	98.85
200	0.6	0.3	0.3	55	22.00	10.15	0.60	100.98

$$f_1(0) = \frac{X - \dfrac{a_1 X + \lambda E X}{P_1(a)}}{E(X,a) - 1} + \frac{a_1 + \lambda E + a_0 P_1(a)}{P_0(a)P_1(a)} \tag{4.31}$$

As in the case of the polynomial (4.21), for the polynomial (4.27), necessary or sufficient conditions for an optimal E,X-policy to exist cannot be obtained. Quantitative results are determined in a similar fashion to that described in Section 4.3b.

The results obtained for the polynomial (4.27) are similar to that obtained for the polynomial (4.21). One notable difference does occur in the results obtained for these polynomials. For a particular curve p(E) and a set of critical variables, the amount of partial repair corresponding to the polynomial (4.27) is less than or equal to the amount of partial repair corresponding to the polynomial (4.21). In addition, there is a decrease in the replacement age X, and an increase in $f_1(0)$.

Table 4.4 illustrates the sensitivity of the parameters for the E,X-policy for a concave curve p(E), as in Figure 4.6. For a concave curve p(E) as in Figure 4.3, Table 4.2 illustrates the change from major repair to minimal repair as the variables are varied one at a time. Comparing the results of Tables 4.2 and 4.3, it is observed that the switch from major to minimal repair occurs at a lower scaling factor for the polynomial (4.27) than for the polynomial (4.21).

Polynomials (4.21) and (4.27) have the same operating cost $g(x) = a_1 x + a_0$ and differ in the expenditure repair function $C(E,u)$ (i.e., $C(E,u) = E$ and $C(E,u) = Eu$), which in general is

$$C(E,u) = Eu^m , \qquad m = 0,1,2,\ldots \tag{4.32}$$

Thus, the amount of partial repair is a non-increasing function of the integer m.

Table 4.4. Table of Optimum Values for E,K-policy for a Convex Curve p(E) as in Figure 4.6

					$\bar{q}(x) = 10x + \lambda Ex$			
K	[illegible]	λ	$\lambda \bar{x}_p(1)$	Scaling Factor	E	N	a	$f_1(0)$
100	0.6	0.1	0.1	30	2.25	6.87	0.75	38.07
100	0.6	0.3	0.3	30	4.20	5.46	0.65	85.63
200	0.2	0.1	0.1	35	15.40	6.78	0.35	496.83
200	0.6	0.1	0.1	35	4.80	8.38	0.65	100.30
300	0.2	0.1	0.1	40	17.60	8.54	0.35	556.37
400	0.2	0.1	0.1	40	23.00	10.18	0.25	587.75
400	0.6	0.3	0.3	20	8.80	17.33	0.35	85.11
400	0.6	0.3	0.3	25	8.12	16.63	0.45	89.15
300	0.6	0.3	0.3	30	6.75	12.55	0.55	82.05
300	0.6	0.45	0.3	30	3.45	12.04	0.70	97.36

Table 4.5 Table of Optimum Values for E,X-policy: Concave Curve p(E) as in Figure 4.5

					$q(x) = 10x + \lambda Ex$			
K	θ	λ	$\lambda \bar{h}_p(t)$	Scaling Factor	E	X	a	$f_1(0)$
100	0.4	0.1	0.1	20	20.00	5.56	0.00	89.20
100	0.4	0.1	0.1	25	0.00	5.74	1.00	91.24
100	0.4	0.3	0.3	25	25.00	4.35	0.00	360.59
100	0.4	0.3	0.3	30	0.00	4.83	1.00	383.18
100	0.6	0.3	0.3	20	20.00	5.35	0.00	85.30
100	0.6	0.3	0.3	25	0.00	5.74	1.00	91.24
200	0.6	0.3	0.3	25	25.00	8.51	0.00	98.63
200	0.6	0.3	0.3	30	0.00	9.12	1.00	105.90
200	0.6	0.45	0.3	15	15.00	8.82	0.00	92.59
200	0.6	0.45	0.3	20	0.00	9.12	1.00	103.90

CHAPTER 5

THE NUMBER OF OPERATING PERIODS IN [0,X] FOR THE LINEAR HYSTERESIS PROBLEM

5.1 Introduction

This chapter investigates a renewal process associated with the linear hysteresis problem. Between successive replacements of the equipment, as it ages from 0 to X, there exist alternating operating and repair periods. Operating periods are terminated by failure, not replacement. At the end of a repair period, there is a partial reduction of the service age of the equipment. Thus, the termination of a repair acts as a point of partial regeneration of the service age process associated with the linear hysteresis problem. At the conclusion of the last repair period prior to replacement, the equipment ages continuously, without failure, until replacement occurs at service age X. The number of operating periods in [0, X], N(X), forms a counting or renewal process. The importance of the renewal process N(X) can be understood in an industrial setting. A plant manager has a piece of equipment which when it fails, prior to replacement, has a major repair performed on it. He is contemplating reducing the level of repair; however, he is not certain what the corresponding effect will be on the expected number of equipment failures prior to replacement.

In this chapter, the probability mass function of N(X) is determined for the linear hysteresis problem with constant failure rate function. In addition, the expected value of N(X) is obtained.

5.2 Probability Mass Function of N(X)

Consider a stochastic process in which the equipment ages continuously while operating and is subject to failure. Upon failure, it is repaired and returned to an operating state with a service age equal to a ($0 \leq a \leq 1$) times the service age at the instant of failure. When an equipment attains a service age X, it is instantly replaced with an identical new (service age zero) equipment. The stochastic process, just described, is the service age process for the linear hysteresis problem.

Define:

$P(N(X) = r)$ = probability that $N(X)$, the number of operating periods in $[0,X]$, is r, $r=0,1,\ldots$

W_i = equipment service age at the commencement of the ith repair, $i=1,2,\ldots$

Let $\{\chi_i\}$, $i=1,2,\ldots$, be a sequence of independently and identically distributed random variables denoting the successive operating times to failure of the equipment whose distribution function is the negative exponential with parameter λ.

Since W_i is the equipment service age at the commencement of the ith repair, then for i=1

$$W_1 = \chi_1 \qquad (5.1)$$

and for i=2

$$W_2 = a\chi_1 + \chi_2$$
$$= aW_1 + \chi_2$$

and in general

$$W_{i+1} = aW_i + \chi_{i+1}, \quad i=1,2,\ldots \qquad (5.2)$$

Solving the difference equation (5.2) subject to the boundary condition (5.1), the following is obtained

$$W_i = \sum_{j=1}^{i} a^{i-j} X_j \quad , \qquad i=1,2,\ldots \tag{5.3}$$

The probability that the number of operating periods in $[0,X]$ is r is equal to the probability that $W_{r+1} > X$ and $W_i < X$ for $i=1,\ldots,r$. This is stated as

$$P(N(X) = r) = P(W_{r+1} > X, W_r < X, W_{r-1} < X, \ldots, W_1 < X) \tag{5.4}$$

Before proceeding to obtain $P(N(X) = r)$ for some a, where $0 \leq a \leq 1$, the boundary cases of $a = 1$ (minimal repair) and $a = 0$ (major repair) are examined.

For $a = 1$ in (5.3)

$$W_i = \sum_{j=1}^{i} X_j \quad , \quad i=1,2,\ldots$$

Since the random variables denoting the operating times to failure are independently and identically distributed as the negative exponential with parameter λ, then the number of operating periods in $[0,X]$ is Poisson distributed and

$$P(N(X) = r) = \frac{(\lambda X)^r e^{-\lambda X}}{r!} \tag{5.5}$$

For $a = 0$ in (5.3)

$$W_i = X_i \quad , \quad i=1,2,\ldots$$

Using this expression in (5.4) yields

$$P(N(X) = r) = P(X_{r+1} > X,\ X_r < X,\ X_{r-1} < X, \ldots, X_1 < X)$$

Since the $\{X_i\}$ are independently and identically distributed random variables with negative exponential distribution, then

$$P(N(X) = r) = P(X_1 > X)[P(X_1 < X)]^r$$

$$= \int_X^{\infty} \lambda e^{-\lambda x} dx \left[\int_0^X \lambda e^{-\lambda x} dx \right]^r$$

$$= e^{-\lambda X}[1 - e^{-\lambda X}]^r, \quad r=0,1,2,\ldots \qquad (5.6)$$

Thus, the probability mass function N(X) is the geometric distribution, where

$$p = e^{-\lambda X} \quad \text{and} \quad q = 1 - e^{-\lambda X}$$

Consider now the probability mass function for N(X) for the linear hysteresis problem, where a is some value, such that $0 \leq a \leq 1$. For r = 0,1,2,3, and 4, P(N(X) = r) is determined. From the values obtained for these probabilities, a general expression for P(N(X) = r) is inferred. Figures 5.1 through 5.4 are sample functions of the service age process for N(X) = 1,...,4, respectively.

Now

$$P(N(X) = 0) = P(W_1 > X)$$

$$= P(X_1 > X)$$

$$= e^{-\lambda X} \qquad (5.7)$$

Note that this is the probability of zero operating periods for major repair. Continuing,

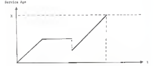

Figure 5.1. Sample function of the service age process $S(t)$ when $N(X) = 1$ for the linear hysteresis problem with downtime and constant failure rate function

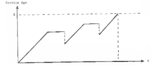

Figure 5.2. Sample function of the service age process $S(t)$ when $N(X) = 2$ for the linear hysteresis problem with downtime and constant failure rate function.

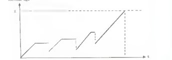

Figure 5.3. Sample function of the service age process $S(t)$ when $N(X) = 3$ for the linear hysteresis problem with downtime and constant failure rate function

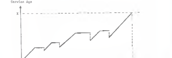

Figure 5.4 Sample function of the service age process $S(t)$ when $N(X) = 4$ for the linear hysteresis problem with downtime and constant failure rate function.

$$P\{N(X) = 1\} = P\{W_2 > X, W_1 < X\}$$

$$= P\{W_2 > X | W_1 < X\}P\{W_1 < X\}$$

$$= P\{aX_1 + X_2 > X | X_1 < X\}P\{X_1 < X\}$$

$$= \int_0^X P(W_2 > X | W_1 = x_1)f_{X_1}(x_1)dx_1$$

$$= \int_0^X P(ax_1 + X_2 > X | W_1 = x_1)f_{X_1}(x_1)dx_1$$

$$= \int_0^X P(X_2 > X - ax_1)f_{X_1}(x_1)dx_1$$

$$P\{N(X) = 1\} = \int_0^X e^{-\lambda(X-ax_1)} \lambda e^{-\lambda x_1} dx_1$$

$$= e^{-\lambda X} \int_0^X \lambda e^{-\lambda x_1(1-a)} dx_1 \tag{5.8}$$

$$P\{N(X) = 1\} = \frac{e^{-\lambda X}}{(1-a)} [1 - e^{-\lambda X(1-a)}] \tag{5.9}$$

Now

$$P\{N(X) = 2\} = P\{W_3 > X, W_2 < X, W_1 < X\}$$

$$= P\{W_3 > X | W_2 < X, W_1 < X\}P\{W_2 < X | W_1 < X\}P\{W_1 < X\}$$

$$= P\{a^2X_1 + aX_2 + X_3 > X | aX_1 + X_2 < X, X_1 < X\}$$

$$P\{aX_1 + X_2 < X | X_1 < X\}P\{X_1 < X\}$$

$$= \int_0^X \int_0^{X-ax_1} P\{X_3 > X - a^2x_1 - ax_2\}f_{X_1}(x_1)f_{X_2}(x_2)dx_1dx_2$$

Or

$$P\{N(X) = 2\} = \int_0^X \int_0^{X-ax_1} e^{-\lambda(X-a^2x_1-ax_2)} \lambda e^{-\lambda x_1} \lambda e^{-\lambda x_2} dx_1 dx_2$$

$$= e^{-\lambda X} \int_0^X \lambda e^{-\lambda x_1(1-a^2)} \int_0^{X-ax_1} \lambda e^{-\lambda x_2(1-a)} dx_1 dx_2 \qquad (5.10)$$

$$= \frac{e^{-\lambda X}}{(1-a)} \int_0^X \lambda e^{-\lambda x_1(1-a^2)} [1 - e^{-\lambda(X-ax_1)(1-a)}] dx_1$$

$$= \frac{e^{-\lambda X}}{(1-a)} \int_0^X \lambda e^{-\lambda x_1(1-a^2)} dx_1$$

$$- \frac{e^{-\lambda X(1-a)}}{(1-a)} [e^{-\lambda X} \int_0^X \lambda e^{-\lambda x_1(1-a)} dx_1] \qquad (5.11)$$

Using (5.8) in (5.11), the following is obtained

$$P\{N(X) = 2\} = \frac{e^{-\lambda X}}{(1-a)} \int_0^X \lambda e^{-\lambda x_1(1-a^2)} dx_1 - \frac{e^{-\lambda X(1-a)}}{(1-a)} P\{N(X) = 1\}$$

Or

$$P\{N(X) = 2\} = \frac{e^{-\lambda X}}{(1-a)(1-a^2)} [1 - e^{-\lambda X(1-a^2)}]$$

$$- \frac{e^{-\lambda X(1-a)}}{(1-a)} P\{N(X) = 1\} \qquad (5.12)$$

Now

$$P\{N(X) = 3\} = P\{W_4 > X, W_3 < X, W_2 < X, W_1 < X\}$$

$$= \int_0^X \int_0^{X-ax_1} \int_0^{X-a^2x_1-ax_2} \lambda^3 e^{-\lambda(X-a^3x_1-a^2x_2-ax_3)}$$

$$e^{-\lambda(x_1+x_2+x_3)} dx_1 dx_2 dx_3$$

$$P\{N(X) = 3\} = e^{-\lambda X} \int_0^X \lambda e^{-\lambda x_1(1-a^3)} \int_0^{X-ax_1} \lambda e^{-\lambda x_2(1-a^2)} \int_0^{X-a^2x_1-x_2} \lambda e^{-\lambda x_3(1-a)}\, dx_1 dx_2 dx_3 \tag{5.13}$$

$$= \frac{e^{-\lambda X}}{(1-a)} \int_0^X \lambda e^{-\lambda x_1(1-a^3)} \int_0^{X-ax_1} \lambda e^{-\lambda x_2(1-a^2)} [1 - e^{-\lambda(X-a^2x_1-ax_2)(1-a)}] \cdot dx_1 dx_2$$

$$= \frac{e^{-\lambda X}}{(1-a)} \int_0^X \lambda e^{-\lambda x_1(1-a^3)} \int_0^{X-ax_1} \lambda e^{-\lambda x_2(1-a^2)}\, dx_1 dx_2$$

$$- \frac{e^{-\lambda X(1-a)}}{(1-a)} [e^{-\lambda X} \int_0^X \lambda e^{-\lambda x_1(1-a^2)} \int_0^{X-ax_1} \lambda e^{-\lambda x_2(1-a)} dx_1 dx_2] \tag{5.14}$$

Substituting (5.10) in (5.14) yields

$$P\{N(X) = 3\} = \frac{e^{-\lambda X}}{(1-a)} \int_0^X \lambda e^{-\lambda x_1(1-a^3)} \int_0^{X-ax_1} \lambda e^{-\lambda x_2(1-a^2)}\, dx_1 dx_2$$

$$- \frac{e^{-\lambda X(1-a)}}{(1-a)} P\{N(X) = 2\}$$

$$= \frac{e^{-\lambda X}}{(1-a)(1-a^2)} \int_0^X \lambda e^{-\lambda x_1(1-a^3)} [1 - e^{-\lambda(X-ax_1)(1-a^2)}] dx_1$$

$$- \frac{e^{-\lambda X(1-a)}}{(1-a)} P\{N(X) = 2\}$$

$$= \frac{e^{-\lambda X}}{(1-a)(1-a^2)} \int_0^X \lambda e^{-\lambda x_1(1-a^3)} dx_1$$

$$- \frac{e^{-\lambda X(1-a^2)}}{(1-a)(1-a^3)} [e^{\lambda X} \int_0^X \lambda e^{-\lambda x_1(1-a)} dx_1]$$

$$- \frac{e^{-\lambda X(1-a)}}{(1-a)} P(N(X) = 2) \qquad (5.15)$$

Using (5.8) in (5.15), the following is obtained

$$P(N(X) = 3) = \frac{e^{-\lambda X}}{(1-a)(1-a^2)} \int_0^X \lambda e^{-\lambda x_1(1-a^3)} dx_1$$

$$- \frac{e^{-\lambda X(1-a^2)}}{(1-a)(1-a^2)} P(N(X) = 1)$$

$$- \frac{e^{-\lambda X(1-a)}}{(1-a)} P(N(X) = 2)$$

or

$$P(N(X) - 3) = \frac{e^{-\lambda X}}{(1-a)(1-a^2)(1-a^3)} [1 - e^{-\lambda X(1-a^3)}]$$

$$- \frac{e^{-\lambda X(1-a^2)}}{(1-a)(1-a^2)} P(N(X) = 1)$$

$$- \frac{e^{-\lambda X(1-a)}}{(1-a)} P(N(X) = 2) \qquad (5.16)$$

Now

$$P(N(X) = 4) = P(W_5 > X, W_4 < X, W_3 < X, W_2 < X, W_1 < X)$$

$$= e^{-\lambda X} \int_0^X \lambda e^{-\lambda x_1(1-a^4)} \int_0^{X-ax_1} \lambda e^{-\lambda x_2(1-a^3)} \int_0^{X-a^2x_1-ax_2} \lambda e^{-\lambda x_3(1-a^2)}$$

$$\int_0^{X-a^3x_1-a^2x_2-ax_3} \lambda e^{-\lambda x_4(1-a)} dx_1 dx_2 dx_3 dx_4$$

$$- \frac{e^{-\lambda X}}{(1-a)} \int_0^X \lambda e^{-\lambda x_1(1-a^4)} \int_0^{X-ax_1} \lambda e^{-\lambda x_2(1-a^3)} \int_0^{X-a^2x_1-ax_2} \lambda e^{-\lambda x_3(1-a^2)}$$

$$dx_1 dx_2 dx_3$$

$$- \frac{e^{-\lambda X(1-a)}}{(1-a)} [e^{-\lambda X} \int_0^X \lambda e^{-\lambda x_1(1-a^3)} \int_0^{X-ax_1} \lambda e^{-\lambda x_2(1-a^2)}$$

$$\cdot \int_0^{X-a^2x_1-ax_2} \lambda e^{-\lambda x_3(1-a)} dx_1 dx_2 dx_3] \qquad (5.17)$$

Substituting (5.13) into (5.17) yields

$$P\{N(X) = 4\} = \frac{e^{-\lambda X}}{(1-a)} \int_0^X \lambda e^{-\lambda x_1(1-a^4)} \int_0^{X-ax_1} \lambda e^{-\lambda x_2(1-a^2)} \int_0^{X-a^2x_1-ax_2}$$

$$\lambda e^{-\lambda x_3(1-a)} dx_1 dx_2 dx_3$$

$$- \frac{e^{-\lambda X(1-a)}}{(1-a)} P\{N(X) = 3\}$$

$$= \frac{e^{-\lambda X}}{(1-a)(1-a^2)} \int_0^X \lambda e^{-\lambda x_1(1-a^4)} \int_0^{X-ax_1} \lambda e^{-\lambda x_2(1-a^3)} dx_1 dx_2$$

$$- \frac{e^{-\lambda X(1-a^2)}}{(1-a)(1-a^2)} [e^{-\lambda X} \int_0^X \lambda e^{-\lambda x_1(1-a^2)} \int_0^{X-ax_1} \lambda e^{-\lambda x_2(1-a)} dx_1 dx_2]$$

$$- \frac{e^{-\lambda X(1-a)}}{(1-a)} P\{N(X) = 3\} \qquad (5.18)$$

Substituting (5.10) in (5.18) and integrating the first term on the right-hand side of (5.18), the following is obtained

$$P\{N(X) = 4\} = \frac{e^{-\lambda X}}{(1-a)(1-a^2)(1-a^3)} \int_0^X \lambda e^{-\lambda x_1(1-a^4)} dx_1$$

$$\frac{e^{-\lambda X(1-a^3)}}{(1-a)(1-a^2)(1-a^3)} [e^{-\lambda X} \int_s^X \lambda e^{-\lambda x_1(1-a)} dx_1]$$

$$- \frac{e^{-\lambda X(1-a^2)}}{(1-a)(1-a^2)} P\{N(X) = 2\} - \frac{e^{-\lambda X(1-a)}}{(1-a)} P\{N(X) = 3\} \tag{5.19}$$

Using (5.8) in (5.19) and integrating the first term on the right-hand side of (5.19) yields

$$P\{N(X) = 4\} = \frac{e^{-\lambda X}}{(1-a)(1-a^2)(1-a^3)(1-a^4)} [1 - e^{-\lambda X(1-a^4)}]$$

$$- \frac{e^{-\lambda X(1-a^3)}}{(1-a)(1-a^2)(1-a^3)} P\{N(X) = 1\}$$

$$- \frac{e^{-\lambda X(1-a^2)}}{(1-a)(1-a^2)} P\{N(X) = 2\}$$

$$- \frac{e^{-\lambda X(1-a)}}{(1-a)} P\{N(X) = 3\} \tag{5.20}$$

By induction or otherwise, the following is obtained

$$P\{N(X) = r\} = \frac{e^{-\lambda X}}{\prod_{j=1}^{r} (1-a^j)} [1 - e^{-\lambda X(1-a^r)}]$$

$$- \sum_{k=1}^{r-1} \frac{e^{-\lambda X(1-a^{r-k})}}{\prod_{j=1}^{r-k} (1-a^j)} P\{N(X) = k\} \qquad r=2,3,\ldots$$

where

$$P(N(X) = 1) = \frac{e^{-\lambda X}}{(1-a)} [1 - e^{-\lambda X(1-a)}]$$

and (5.21)

$$P(N(X) = 0) = e^{-\lambda X}$$

Expression (5.21) is the probability mass function of the number of operating periods in [0,X]. For notational convenience, let

$$P_i = P(N(X) = i) \tag{5.22}$$

$$u_i = \frac{e^{-\lambda X}}{\prod_{j=1}^{i} (1-a^j)} [1 - e^{-\lambda X(1-a^i)}] \tag{5.23}$$

$$g_i = \frac{e^{-\lambda X(1-a^i)}}{\prod_{j=1}^{i} (1-a^j)} \tag{5.24}$$

Using (5.22), (5.23), and (5.24) in (5.21), then the following is obtained

$$P_r = \begin{cases} e^{-\lambda X} & r = 0 \\ u_1 & r = 1 \\ u_r - \sum_{m=1}^{r-1} g_{r-m} P_m & r = 2, 3, \ldots \end{cases} \tag{5.25}$$

This compact form for expressing $P(N(X) = r)$ is used in the following section to obtain the expected value of $N(X)$, $E[N(x)]$.

5.3 Expected Value of N(X)

By definition

$$E[N(X)] = \sum_{r=0}^{\infty} r\, P(N(X) = r)$$

Using (5.22) in the last expression, the following is obtained

$$E[N(X)] = \sum_{r=0}^{\infty} rP_r$$

$$= \sum_{r=1}^{\infty} rP_r \tag{5.26}$$

$$= P_1 + \sum_{r=2}^{\infty} rP_r \tag{5.27}$$

Substituting (5.25) in (5.27) yields

$$E[N(X)] = a_1 + \sum_{r=2}^{\infty} r[a_r - \sum_{m=1}^{r-1} \beta_{r-m}P_m]$$

$$= a_1 + \sum_{r=2}^{\infty} ra_r - \sum_{r=2}^{\infty} \sum_{m=1}^{r-1} r\beta_{r-m}P_m$$

$$= \sum_{r=1}^{\infty} ra_r - \sum_{r=2}^{\infty} \sum_{m=1}^{r-1} (r - m + m)\beta_{r-m}P_m$$

$$= \sum_{r=1}^{\infty} ra_r - \sum_{r=2}^{\infty} \sum_{m=1}^{r-1} (r - m)\beta_{r-m}P_m$$

$$- \sum_{r=2}^{\infty} \sum_{m=1}^{r-1} m\beta_{r-m}P_m \tag{5.28}$$

Expanding $\sum_{r=2}^{\infty} \sum_{m=1}^{r-1} (r - m)\beta_{r-m}P_m$, it is observed that

$$\sum_{r=2}^{\infty} \sum_{m=1}^{r-1} (r - m)\beta_{r-m}P_m$$

$$= \beta_1 P_1$$

$$+ \beta_1 P_2 + 2\beta_2 P_1$$

$$+ \beta_1 P_3 + 2\beta_2 P_2 + 3\beta_3 P_1$$

$$+ \beta_1 P_4 + 2\beta_2 P_3 + 3\beta_3 P_2 + 4\beta_4 P_1$$

$$+ \quad \cdot \cdot$$

$$= \sum_{r=1}^{\infty} r\beta_r \sum_{i=1}^{\infty} P_i \tag{5.29}$$

Similarly, expanding $\sum_{r=2}^{\infty} \sum_{m=1}^{r-1} m\beta_{r-m} P_m$

$$\sum_{r=2}^{\infty} \sum_{m=1}^{r-1} m\beta_{r-m} P_m$$

$$= \beta_1 P_1$$

$$+ 2\beta_1 P_2 + \beta_2 P_1$$

$$+ 3\beta_1 P_3 + 2\beta_2 P_2 + \beta_3 P_1$$

$$+ 4\beta_1 P_4 + 3\beta_2 P_3 + 2\beta_3 P_2 + \beta_4 P_1$$

$$+ \cdots$$

$$= \sum_{r=1}^{\infty} \beta_r \sum_{i=1}^{\infty} iP_i \tag{5.30}$$

Substituting (5.29) and (5.30) in (5.28), yields

$$E[N(X)] = \sum_{r=1}^{\infty} r\pi_r - \sum_{r=1}^{\infty} r\beta_r \sum_{i=1}^{\infty} P_i - \sum_{r=1}^{\infty} \beta_r \sum_{i=1}^{\infty} iP_i$$

Using (5.26) in the last expression and since $\sum_{i=0}^{\infty} P_i = 1$, the following

is obtained

$$E[N(X)] = \sum_{r=1}^{\infty} ra_r - \sum_{r=1}^{\infty} r\beta_r(1 - F_0) - \sum_{r=1}^{\infty} \beta_r E[N(X)]$$

Collecting terms in $E[N(X)]$ and since $F_0 = e^{-\lambda X}$, then

$$(1 + \sum_{r=1}^{\infty} \beta_r)E[N(X)] = \sum_{r=1}^{\infty} ra_r - \sum_{r=1}^{\infty} r\beta_r(1 - e^{-\lambda X})$$

$$= \sum_{r=1}^{\infty} r(a_r + e^{-\lambda X}\beta_r - \beta_r)$$

or

$$E[N(X)] = \frac{\sum_{r=1}^{\infty} r(a_r + e^{-\lambda X}\beta_r - \beta_r)}{1 + \sum_{r=1}^{\infty} \beta_r} \qquad (5\ 31)$$

Expression (5 31) permits the computation of the expected number of operating periods in $[0,X]$ for a given value of a.

CHAPTER 6

CONCLUSIONS AND RECOMMENDATIONS

6.1 Conclusions

This dissertation introduced to replacement theory the notion of partial repair of a failed equipment and the subsequent partial reduction (or hysteresis) of service age. By varying the amount of partial reduction of service age, the problems of minimal repair and major repair could be recovered. It was assumed that the equipment ages continuously and is subject to failure, where failures are detected instantly. Failure resulted in a partial repair being performed for a length of time, called the downtime for repair. Upon attaining a predetermined service age X, the equipment was replaced instantaneously with an identical new one. The repair and replacement problem generated under these assumptions was the hysteresis problem. A cost structure incorporating variable operating costs, repair costs, replacement costs, and loss of revenue per unit of downtime was assumed.

Using an optimization approach known as the functional equation approach, an expression was obtained for the minimum total expected discounted cost over an infinite time horizon. For the special case where the hysteresis age function recovered all the service life (i.e., major repair), a sufficient condition for a unique finite optimal replacement age X was determined. This condition was an extension to prior results obtained for the major repair problem. The significant contributions to this problem were the incorporation of downtime for repair and variable repair costs.

When the hysteresis age function was linear and the failure rate function was constant, a pair of sufficient conditions for a unique finite optimal equipment replacement age X to exist were obtained. One of the conditions concerned the shape of the cost function $g(x)$. It was determined that the polynomial form representing $g(x)$ must have all positive coefficients. Therefore, $g(x)$ was a convex function. Numerical results which illustrated the value of X and the corresponding cost for following an optimal X-policy were tabulated for two polynomials $g(x)$, as several of the critical variables were varied. The results of the sensitivity analysis on the tabulated results are contained in Chapter 3.

An extension to the linear hysteresis problem for which the amount of partial repair was economically controlled required the addition of a new decision variable. Analytic results for the resulting two variable decision policy were not possible. However, analysis of the quantitative results obtained indicate that for certain service age recovery functions $p(E)$, only minimal repair at major repair was performed. In addition, the amount of partial repair performed for $C(E,u) = E$ was greater than or equal to the amount of partial repair performed for $C(E,u) = Eu$.

A probability mass function for the number of operating periods between successive replacements, $N(X)$, was obtained. Also, the expected value of $N(X)$ was determined.

6.2 Recommendations for Future Research

Several interesting possibilities for future research are indicated by the results developed in this dissertation.

Comparing the results obtained by Sivazlian [21] for the minimal repair problem and for the major repair problem of Chapter 2, it would be of interest to solve the linear hysteresis problem without the restriction of specifying the failure rate function. This would require a different approach to obtaining the general solution $f_1(x)$ than that considered in Chapter 3. Another possibility is to consider a non-constant failure rate function for the linear hysteresis problem. However, the assumption of a series solution does not lead to an answer in that case.

Recall the functional-differential equation (1.10)

$$f_1'(x) - [1 + \lambda(x)]f_1(x) - \lambda(x)\overline{h}_T(1)f_1(a(x)) = - g(x)$$

The only hysteresis age function $a(x)$ that has been investigated for this equation has been linear (i.e., $a(x) = ax$). A different function $a(x)$ could be examined and the solution obtained to the resulting functional-differential equation.

More attention needs to be spent on the economically controlled linear hysteresis problem. Although a number of curves $p(E)$ have been examined for two expenditure repair functions $C(E,x)$, only a linear cost function $g(x)$ has been investigated. The areas of future research in this problem encompass considering different curves $p(E)$ and other combinations of $\hat{g}(x)$ and $C(E,x)$. Since analytic results cannot be obtained, it would be of interest to develop a better search procedure to determine the optimum values of E,X for the economically controlled linear hysteresis problem.

There are some interesting stochastic processes generated by the hysteresis problem which are areas for future research. The distribution of the number of equipment replacements in a finite length of time is one possible area. Another possible area is the transient and steady state probabilities of equipment age

APPENDICES

APPENDIX A

Consider the equations

$$f_1'(x) - A(x)f_1(x) + B(x)f_1(s(x)) = 0 \qquad (A\ 1)$$

$$f_1'(x) - A(x)f_1(x) + B(x)f_1(s(x)) = C(x) \qquad (A\ 2)$$

The properties of $f_1(x)$ are continuity, monotone increasing in x, and existence of the first derivative. The functions A(x) and B(x) are continuous in x. C(x) is a known function of x and s(x) is a bounded and continuous function of x. The following theorem is now proved.

Theorem A.1

If $f_1(x) = y_1(x)$ is any solution (called the complementary solution) to the homogeneous equation (A 1) and $f_1(x) = y_2(x)$ is any solution (called the particular solution) to the non-homogeneous equation (A 2), then $f_1(x) = y_1(x) + y_2(x)$ is a solution to (A 2).

Proof

Since $y_1(x)$ is a solution to (A 1), the following holds

$$y_1'(x) - A(x)y_1(x) + B(x)y_1(s(x)) = 0 \qquad (A\ 3)$$

Further since $y_2(x)$ is a solution to (A 2)

$$y_2'(x) - A(x)y_2(x) + B(x)y_2(s(x)) = C(x) \qquad (A\ 4)$$

Adding (A.3) and (A.4) and recalling that the derivative of a sum is the sum of the derivatives, the following is obtained

$$\frac{d}{dx}[y_1(x) + y_2(x)] + A(x)[y_1(x) + y_2(x)]$$

$$+ B(x)[y_1(s(x)) + y_2(s(x))] = C(x)$$

which is an alternative way of stating that $f_1(x) = y_1(x) + y_2(x)$ is a solution. Thus, the general solution to (A 2) is the sum of the complementary and particular solutions.

A sufficient condition for the existence and uniqueness of the general solution $f_1(x)$ is given by the following theorem.

Theorem A.2

Let $A(x)$, $B(x)$, and $C(x)$ be functions which are continuous in the interval $0 \leq x \leq X$ and $s(x)$ is continuous in the interval $0 \leq x \leq X$. Then there exists a solution $f_1(x)$ satisfying the differential equation (A 2) and also the condition

$$f_1(X) = K(X) + f_1(0)$$

Furthermore, this solution is unique.

Obtaining a complementary solution to (A 1) is not a trivial task. Consider the special case where the hysteresis age function, $s(x)$, equals qx. In 1936, Gregory [15] examined solutions to equations of the form $f(x) = bf(qx) + c$ $(q > 0,\ q \neq 1)$, which he labeled a q-difference equation. His method of solution for q-difference equations has had little application to differential q-difference equations. Efforts to apply his method to obtain a complementary solution to (A.1) when $s(x) = qx$ have met with no success.

One extremely powerful method of differential equations known

as the method of integration in series has resulted in a complementary solution being obtained for the case $a(x) = ax$, $\lambda(x) = 1$. The complementary solution is assumed to be of the form

$$y_1(x) = \sum_{j=0}^{\infty} a_j x^j$$

Upon appropriately substituting the assumed form of the solution in the homogeneous equation, the values of the constants a_j are determined in terms of a_0, an arbitrary constant.

APPENDIX B

Consider the power series

$$S(x,\theta,\lambda\bar{h}_T(i),a) = 1 + \sum_{j=1}^{\infty} \frac{x^j}{j!} \prod_{k=0}^{j-1} (\theta - \lambda\bar{h}_T(i)a^k) \qquad \text{(B.1)}$$

Applying the ratio test for power series of the form $\Sigma\, a_n x^n$ it is determined that

$$R = \lim_{j\to\infty} \left| \frac{\frac{1}{j!} \prod_{k=0}^{j-1} (\theta - \lambda\bar{h}_T(i)a^k)}{\frac{1}{(j+1)!} \prod_{k=0}^{j} (\theta - \lambda\bar{h}_T(i)a^k)} \right|$$

$$= \lim_{j\to\infty} \left| \frac{j+1}{\theta - \lambda\bar{h}_T(i)a^j} \right|$$

$$= \infty$$

where R is the radius of convergence of the power series. Therefore for any value of a less than infinity, the power series $S(x,\theta,\lambda\bar{h}_T(i),a)$ converges.

Taking the first derivative of $S(x,\theta,\lambda\bar{h}_T(i),a)$ with respect to x, the following is obtained

$$\frac{dS}{dx} = (\theta - \lambda\bar{h}_T(i)) + \sum_{j=1}^{\infty} \frac{x^j}{j!} \prod_{k=0}^{j} (\theta - \lambda\bar{h}_T(i)a^k) \qquad \text{(B.2)}$$

Further, the nth derivative of $S(x,\theta,\lambda\bar{h}_T(i),a)$ with respect to x is given by

$$\frac{d^n S}{dx^n} = \prod_{k=0}^{n-1} (\theta - \lambda\bar{h}_T(i)a^k) + \sum_{j=1}^{\infty} \frac{x^j}{j!} \prod_{k=0}^{j+n-1} (\theta - \lambda\bar{h}_T(i)a^k) ,$$

$$n \geq 1 \qquad \text{(B.3)}$$

The radius of convergence of $\frac{d^n S}{dx^n}$ $(n \geq 1)$ is infinity.

For x a positive real, $[S(x,\theta,\lambda\bar{h}_T(i),a) - 1]$ is positive. If x is a non-negative real, then $S(x,\theta,\lambda\bar{h}_T(i),a)$ and all its derivatives with respect to x are positive. In addition, the derivatives exist.

The function $S(x,\theta,\lambda\bar{h}_T(i),a)$ may be transformed to $S(Y,B,A)$ by the following sequence of transformations.

$$S(x,\theta,\lambda\bar{h}_T(i),a) = 1 + \sum_{j=1}^{\infty} \frac{x^j}{j!} \prod_{k=0}^{j-1} (\theta - \lambda\bar{h}_T(i)a^k)$$

$$= 1 + \sum_{j=1}^{\infty} \frac{(\theta x)^j}{j!} \prod_{k=0}^{j-1} \left(1 - \frac{\lambda\bar{h}_T(i)}{\theta} a^k\right) \qquad \text{(B.4)}$$

Let

$$Y = \theta x$$

$$B = \frac{\lambda\bar{h}_T(i)}{\theta}$$

$$A = a$$

Then

$$S(Y,B,A) = S(x,\theta,\lambda\bar{h}_T(i),a)$$

$$= 1 + \sum_{j=1}^{\infty} \frac{Y^j}{j!} \prod_{k=0}^{j-1} (1 - BA^k) \qquad \text{(B.5)}$$

For various values of the parameters Y, B, and A, the value of the function $S(Y,B,A)$ is displayed in Tables B.1 through B.9. The parameter B is varied from 0.1 to 0.9 in increments of 0.1. For each

of B's nine values, the parameters Y and A are varied [illegible] in increments of 0.2 and from 0 to 1 in increments of 0.1 [illegible]. The function $\mathrm{Log}_{10}\ S(Y,B,A)$ is plotted against the parameter Y in Figures B 1 through B 9. In each figure for a particular B, eleven curves are plotted corresponding to values the parameter A can assume. Whenever space permits, the curves are labeled in the right margin with the appropriate value of A. The unlabeled curves follow the rule that a lower curve has a higher value of the parameter A than a higher curve.

As can be observed from the tables and figures, $S(Y,B,A)$ increases as Y increases. However, $S(Y,B,A)$ decreases when either A or B increases. In terms of the original function $S(x,\theta,\lambda\bar{h}_T(i),s)$, the value of $S(x,\theta,\lambda\bar{h}_T(i),s)$ increases when either θ or x increases, and decreases when either $\lambda\bar{h}_T(i)$ or s increases.

Table 3.1. Table of $S(Y,B,A)$ for $B = 10$

[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
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[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]

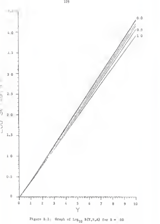

Figure B.1. Graph of Log_{10} B(Y,B,A) for B = 10

Table B 2   Table of Ψ(τ,β,A) for B = .20

τ	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]

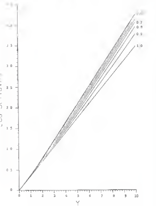

Figure 5.2 Graph of Log_{10} S(Y,B,A) for B = .20

Table 8.3 Table of S(T,B,A) for B = 30

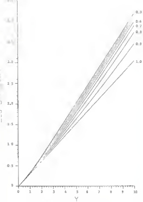

Figure 8-3 Graph of Log_{10} S(Y,B,A) for B = .30

Table B.4. Table of S(Y,h,1) for h [illegible]

[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]

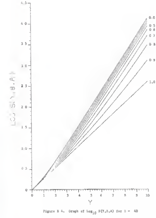

Figure B 4. Graph of $\mathrm{Log}_{10}\ S(Y,B,A)$ for B = 40

Table 6.5 Table of $S(Y, B, A)$ for $B =$ [illegible]

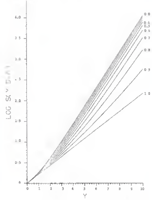

Figure B.5 Graph of Log_{10} S(Y,B,A) for B = .50

Table 8.6. Table of S([illegible]) for n = 60

	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]

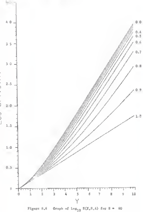

Figure 8.6 Graph of Log_{10} S(Y,B,A) for B = 60

Table B.7 Table of $S(Y, Y, A)$ for B = 70

Y	0.0	0.10	0.20	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]

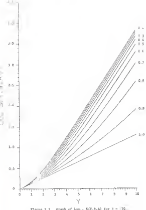

Figure 3 7 Graph of Log_{10} S(Y,B,A) for B = .70

Table 5.8 Table of S(Y, N, A) for N = 80

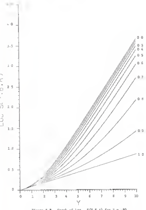

Figure 8 B. Graph of $\log_{10} S(Y,B,A)$ for B = .60

Table 8.9. Table of $S(Y_s, B_s, A)$ for $B = 90$

[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]

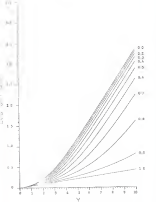

Figure 8.5. Graph of Log_{10} S(Y,B,A) for B = 90

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